A STUDY ON THE STRUCTURE OF GLOBULAR CLUSTERS

N. Owaki

Received March 20, 1961

Abstract

It is aimed at to investigate changes of structures of globular clusters with time, and to examine whether they are actually in non-equilibrium states. In Chapter I, a model process for energy exchange is considered instead of treating the Fokker-Planck equation, and an approximate equation for distribution of velocity is derived. The distribution function Ψ of total mechanical energy, H, and angular momentum, J, of a star is considered to be more reasonable for studying globular clusters rather than distribution function of velocity itself. An equation for Ψ is derived. It is found that the rate of escape of stars from a spherical cluster can be expressed in a similar form to that of an infinitely extended homogeneous cluster, and it is determined only by the mean of local rates of escape. On the basis of an assumption that the initial cluster was restricted in a finite region with a finite mass, the possibility of structure change and expansion is suggested, and it is seen for the cluster to tend to equilibrium asymptotically as $t \to \infty$. In Chapter II, timely change of structure of a model cluster of stars with an equal mass and of uniform rate of energy exchange is investigated. It is shown in this case that the distribution of stars changes accompanying envelope formation from its initial one to a truncated isothermal distribution having a finite mass and radius. The computed result agrees considerably well with observations. The change of radius of a cluster with time is also discussed, and it is found to increase at early stages of its evolution. It is concluded: (1) large clusters, e.g., ω Cen might be still in non-equilibrium states, and its peculiar distribution could be explained only as in an non-equilibrium state, (2) medium size clusters like M3 or M15 are close to equilibrium but have not completely reached it, (3) the clusters are generally still in expanding stages. The ages of the clusters are estimated from the characteristics of density distribution. For M2, M3, M5, M13, M15, M22, M92 and ω Cen, the results are 8.1, 6.5, 11, 6.6, 6.2, 8.3, 4.6 and 5.1 in 10^9 years, respectively. The results are generally fairly coincident with those obtained from the theory of stellar evolution.

I. Approximate Method of Solution for Velocity Distribution.

1. Introduction.

The structure of globular clusters has been studied since Plummer(1911) and von Zeipel(1913). They assumed the clusters as polytropic spheres in which the member stars were considered as molecules. Afterwards Eddington(1915) assumed the cluster as a statinary assembly of stars of which velocities followed the ellipsoidal distribution law. He derived the internal density distribution of the cluster, but it turned out that the total mass of the cluster became infinite. Woolley(1954), Woolley and Robertson(1956) and von Hoerner(1956) pointed out that there should arise this difficulty so far as the clusters are treated in the analogy of gas sphere, and they put their investigation on the basis of the theory of relaxation.

The fundamental hypothesis of their investigations is that the clusters are in equilibrium. However, if in equilibrium, the velocity distribution of the member stars becomes Maxwellian and thus the internal structure takes the form of isothermal configuration, and consequently, the whole mass of the cluster becomes infinite. To avoid this difficulty, it is necessary to introduce particular assumptions for the velocity distribution, $f(u, v, w, r)$, or to introduce cutoffs for the total mechanical energy of a star, H, and its angular momentum around the center of the cluster, J. The former cutoff is inevitable in any theory of the structure, but the latter seems to have been made rather artificially in the equilibrium theory. Thus it is likely that the structure can be well understood only on the basis of non-equilibrium hypothesis; the both cutoffs can be naturally introduced without any *ad hoc* assumption except for the one concerning the initial configuration and the law of timely change of velocity distribution due to mutual interactions between the member stars.

The purpose of the present paper is to study the structure of the globular clusters to find its timely change, and to conclude that they are still in non-equilibrium state by comparing with observations, and to obtain their ages. The basic hypothesis is that a globular cluster was a spherical assembly of stars of a finite mass and radius when it was formed at $t=0$, and that the velocity distribution f changes from its initial one f_0 as the time goes on, and consequently the escape of stars to outer space takes place and also some of the member stars evaporate beyond the initial radius R_0 to form the outer envelope of the cluster. In Chapter I a model process for change of the distribution function of the member stars in the phase space is given on the basis of two principles (1) that any initial distribution would tend to Maxwellian as the result of mutual interactions of stars and (2) that we shall consider the change of the number of the orbits specified by a set of (H, J), instead of considering velocity distribution function itself at each point in the cluster; this treatment being likely to be more reasonable for the study on escape and evaporation of stars. In Chapter II a simplified cluster model is considered, in which the rate of relaxation is assumed to be uniform throughout the cluster, and timely changes of structure are investigated and the ages of clusters are estimated. In this model only the central parts of a cluster, which might be referred to as core, are considered without taking the outer envelope of the cluster into account, which is formed by evaporation of stars from the core, and consequently the cutoff of the distribution function with respect to J is ignored. However, this simple model gives rather good information for the dynamical evolution of clusters.

As stated before, it has already been indicated by various investigators that to the theory of stellar system hydrodynamical method and the analogy of the gas theory are not available. Kurth has pointed out (1957) that in the hydrodynamical treatment the mutual interaction between a pair of stars is neglected, and therefore that such model could be only useful to a study on the property of a stellar system for a short time interval. On the other hand, the analogy of kinematical theory of gas would not be applicable to stellar system, as stated by Woolley(1954); firstly, length of mean free path of a star in a system is generally of the same order of magnitude as the dimension of the system itself, and secondly, time required for the energy exchange of a star due to mutual encounters to become effective is almost equal to or longer than the period of orbital motion of the star in the system, and finally, the Newtonian force of the mutual interaction is of much longer range than the forces between gas molecules.

Thus the only satisfactory theory seems to have to be built up on the theory of stochastic process of stellar motions through an assembly of stars, as developed by Chandrasekhar(1943*a*, *b*, *c*, *d*), by assuming that the motion is a kind of Brownian motion under fluctuating forces from neighboring stars as well as the smoothed out gravitational forces from the entire assembly. The motion of the star suffers not only from these fluctuaing forces but also from a systematic deceleration, the dynamical friction as their cumulative effect. They are assumed to be governed by a certain probabilistic law, and consequently the change of star's velocity follows a transition probability derived from the law. Thus the theory enables us to compute the probability of finding the star in a volume element of the phase space (x, y, z, u, v, w) at a time t, if the initial position and velocity (r_0, v_0) of the star was given at $t=0$, and it is proved that the probability distribution asymptotically tends to Maxwellian as t tends to infinity, so far as the escape of star from the system does not take place. The probability distribution is given by a transition equation, which can be reduced to a diffusion equation in the velocity space, viz., the Fokker-Planck (Chandrasekhar 1943*a*);

$$\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}+w\frac{\partial f}{\partial z}+K_x\frac{\partial f}{\partial u}+K_y\frac{\partial f}{\partial v}+K_z\frac{\partial f}{\partial w}=$$
$$\mathrm{div}_v\,(k\,\mathrm{grad}_v f)+\mathrm{div}_v\,(\eta v f) \qquad (1)$$

where K is the smoothed out gravitational force, η the coefficient of dynamical friction, k the coefficient of diffusion connected with the mean square velocity

$\overline{|v|^2}$ under the condition that the eq. (1) must be satisfied by Maxwellian identically:

$$\frac{k}{\eta} = \frac{1}{3}\overline{|v|^2}. \tag{2}$$

The initial condition of eq. (1) is of course $f = \delta\ (r - r_0)\ \delta\ (v - v_0)$ at $t = 0$.

From the Boltzmann equation of gas theory a similar equation to eq. (1) can be derived with a different differential form in the right hand side (Rosenbluth, MacDonald and Judd 1957); the eq. (1) is always satisfied by Maxwellian regardless of the behavior of η, but the latter equation is satisfied only for the state when the Boltzmann's H is equal to zero. This seems to come from essential differences between the basic principles underlying the theories of Brownian motion and molecular motion of gas; in the latter, for instance, it is assumed that the exchange of energy and momentum is made locally and that the reverse process of encounter must always occur (e.g., Jeans 1925).

It is, however, extremely difficult to obtain a solution of the equation for general cases with appropriate intial and boundary conditions. Thus this method has so far been only applied to investigate the rate of escape of stars from a modelarized cluster i.e., a uniformly extended and homogeneous cluster by ignoring the space coordinates. It was studied by Chandrasekhar (1943*c*, *d*), White(1949), Spitzer and Härm(1958) and King(1960) following this way, while by Agekyan(1959*a*, *b*) through another method.

In order to study the structure and evolution of stellar clusters, it is the essential problem to clarify the timely change of distribution of velocity as a function of spatial coordinates. Until an elaborate method of solving the Fokker-Planck equations for the problem of structure is established, we cannot help constructing models, for instance, of potential, of velocity or density distributions of clusters for comparison with observation: it was the Woolley's point of view(1954) to follow this way by making models of velocity distribution under some mathematical restrictions on it. Our present way is to progress by considering more fundamental notion on what mechanism is occuring in clusters which must govern the velocity distribution, or in other words by constructing model for dynamical process as the consequence of exchange of mechanical energies between stars.

2. Approximation by Interpolation Formula.

From the property of timely change of the existence probability of stars in the phase space, it is obvious that the distribution function of velocities in the stellar system must be reduced from its initial one to Maxwellian as t tends to infinity. The speed of reduction is determined by the mean time of relaxation of the system, $\overline{T_E}$, the time required for the exchange of energy in the system to become effective. This is in a sense an averaged value of the times of relaxation of individual star T_E which is inversely proportional to the coefficient of dynamical friction η (Chandrasekhar 1943*b*, *c*).

Ambartsumyan(1938), Spitzer(1940) and King(1958*a*) considered a model process for the timely change of velocity distribution function that the distribution remains the initial form when $t < \overline{T_E}$, and after each time of relaxation the Maxwellian is accomplished. Though this is quite an elaborate model, it is rather mathematically inconvenient owing to discontinuity at $t = \overline{T_E}$. we shall then approximate the process with an interpolation formula.

In this study we shall limit ourselves to consider a cluster in which all stars have an equal mass, m. Without loss of generality we may put m equal to unity to simplify calculus.

Let us consider a spatially homogeneous and indefinitely extended stellar cluster having an isotropic velocity distribution, and let $f dudvdw$ be the number of stars of velocities in the range $(\boldsymbol{v}, \boldsymbol{v}+d\boldsymbol{v})$. We shall then assume that the timely change of the velocity distribution be represented by an interpolation formula

$$f(u, v, w, t) = f(u, v, w, o)\, e^{-\beta t} + C e^{-j^2 |v|^2} (1 - e^{-\beta t}), \tag{3}$$

where j and C are parameters to be determined later, and $\beta = 1/\overline{T_E}$, $\overline{T_E}$ being given by Chandrasekhar(1942, eq. (2.379)):

$$\overline{T_E} = \frac{1}{16}\sqrt{\frac{3}{\pi}}\, \frac{\overline{|v|^2}}{G^2 m^2 N \ln\alpha}, \qquad \alpha = \frac{D_0\, \overline{|v|^2}}{2\,Gm}, \tag{4}$$

where N is the number of stars per unit volume, m the mass of stars, D_0 the average distance between stars, and G the gravitational constant.

From eq. (3) we have a differential equation of f with respect to t

$$\frac{\partial f}{\partial t} = -\beta f + \beta C e^{-j^2 |v|^2}, \tag{5}$$

by which we can extend eq. (3) to the case when $\overline{T_E}$, C and j depend on t. Therefore we shall adopt the eq. (5) as the basic equation for our approximation.

The quantities j and C are determined from the conservations of mass and energy. Let H be the kinetic energy of stars per unit volume. Then, for an indefinitely extended homogeneous cluster,

$$\frac{\partial N}{\partial t} = 0 \qquad \text{where} \qquad N = \iiint_{-\infty}^{\infty} f\, dudvdw, \tag{6}$$

and

$$\frac{\partial H}{\partial t}=0 \qquad \text{where} \qquad H=\iiint_{-\infty}^{\infty}\frac{1}{2}\,|\,v\,|^2 f dudvdw \equiv \frac{1}{2}\,N\overline{|\,v\,|^2}. \tag{7}$$

Hence from eq. (5)

$$C=\frac{\iiint_{-\infty}^{\infty} f dudvdw}{\iiint_{-\infty}^{\infty} e^{-j^2|\,v\,|^2} dudvdw}=\frac{j^3 N}{\pi^{\frac{3}{2}}} \qquad \text{and} \qquad \frac{1}{j^2}=\frac{2}{3}\,\overline{|\,v\,|^2}. \tag{8}$$

The process represented by eq. (5) might be interpreted as follows: Let us consider one-dimensional case. Out of $f(v')dv'$ stars with velocities in the range $(v', v'+dv')$, a fraction by an amount $\beta f(v')dv'$ leaves this range per unit time by mutual encounters with other stars and is redistributed over all velocity space to form Maxwellian:

$$\frac{j^3}{\pi^{\frac{3}{2}}}\, e^{-j^2|\,v\,|^2}\, dv.\ \beta f\ (v')\ dv'. \tag{9}$$

Such process occurs everywhere in the whole velocity space, therefore, the change of $f(v)dv$ at a particular velocity v during dt is

$$\frac{\partial}{\partial t}\, f\,(v)\, dv.\, dt = -\,\beta f\,(v)\, dv.dt + \int_{v'=-\infty}^{v'=+\infty}\left(\frac{j^3}{\pi^{\frac{3}{2}}}\, e^{-j^2|\,v\,|^2}\, dv\right)\beta f(v')dv'dt, \tag{10}$$

which is eq. (5).

It may be here remarked that in the case of Brownian motion the stars initially in the range $(v', v'+dv')$ leave the range and take other velocities stepwise and gradually form Maxwellian as a result of diffusion process, while in our model process, some fraction of stars in the range is scattered away and immediately forms Maxwellian without passing through any intermediate stage. This must be the basic difference of our approximation from the real process of the Brownian motion.

3. The Case when Stars escape from the Cluster.

For the case when stars escape from the cluster on arriving at a velocity of escape, v_e we have only to replace C with C_e defined as

$$C_e=\iiint_{|v|<v_e} f dudvdw \Big/ \iiint_{-\infty}^{\infty} e^{-j^2|v|^2}\, dudvdw. \tag{11}$$

In the exact method through Fokker-Planck equation, a boundary condition is imposed on its solution that the solution shall vanish at v_e. Therefore, the velocity distribution deviates from Maxwellian, as shown by Chandrasekhar

(1943*b*), and Spitzer and Härm(1958) indicated that the deviation is greater for smaller masses of stars. However, the present approximation seems to yield fairly good results for general cases.

The density and energy per unit volume expressed as

$$N = \iiint\limits_{|v|<v_e} f dudvdw \qquad \text{and} \qquad H = \iiint\limits_{|v|<v_e} \frac{1}{2} |v|^2 f\, dudvdw \tag{12}$$

decrease at rates λ and η, respectively, which are by eqs. (5) and (11),

$$\lambda = -\frac{1}{N}\frac{dN}{\beta dt} = 1 - 2\Phi_2(x_e), \tag{13}$$

and

$$\mu = -\frac{1}{H}\frac{dH}{\beta dt} = 1 - \frac{4}{3}\Phi_4(x_e), \tag{14}$$

where

$$\Phi_2(x) = \frac{2}{\sqrt{\pi}}\int_0^x x^2 e^{-x^2} dx, \quad \Phi_4(x) = \frac{2}{\sqrt{\pi}}\int_0^x x^4 e^{-x^2} dx, \text{ and } x_e = jv_e. \tag{15}$$

In the above reduction, j has been considered to be given by eq. (8).

The eq. (13) is naturally identical with that by Chandrasekhar(1942, eq. (5.401)). The values of λ and η can be evaluated against x_e by means of tables for $\Phi_2(x)$ and $\Phi_4(x)$ (e.g., Owaki 1960). For the value of x_e equal to $\sqrt{6}$, or 2.4495, which corresponds to the mean value of $v_e/\sqrt{|v|^2}$ equal to 2 introduced by Chandrasekhar, λ ond η becomes equal to 0.00738 and 0.0348, respectively.

4. Extension to the Case involving Space Coordinates.

The approximate equation (5) can be extended to the case where the space coordinates are considered, on the basis of interpretation mentioned at the end of Sec. 2, or from the idea that at each point of a cluster the energy exchange occurs so that local distribution of velocity is to tend to Maxwellian. We must then be concerned with the change of number of stars $f(u, v, w, x, y, z, t)dudvdwdxdydz$ in a small range of phase space along their orbit. Since the rate of energy exchange, or exchange time is a function of local density and mean square velocity, it is generally a function of position. Thus we have an equation

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} + K_x\frac{\partial f}{\partial u} + K_y\frac{\partial f}{\partial v} + K_z\frac{\partial f}{\partial w} = -\beta f$$
$$+ \beta\frac{j^3 e^{-j^2|v|^2}}{\pi^{\frac{3}{2}}}\iiint\limits_{-\infty}^{\infty} f dudvdw. \tag{16}$$

Let H be the total mechanical energy of a star, $\frac{1}{2}|v|^2 + V(x, y, z)$, where

V is the potential of the cluster. Then, if in a stationary state, f must clearly be the Maxwell-Boltzmann distribution, const. $\times$ $\exp(-2j^2H)$ as seen from eq. (16), because its left hand side is to become zero when f is a function only of integral of the motion. Therefore j must be constant throughout the cluster, being not identical with the mean square velocity $\overline{|\boldsymbol{v}|^2}$, which should be a function of position and probably of time. The value of j is determined from the dispersion of the distribution of H, and does not depend on time. And if the initial dispersion at $t=0$ is known, we are to be able to compute the value of j .

On the other hand, if ejection of stars beyond v_e is taken into consideration, the same modification as given in Sec. 3 can be applied to the triple integral in the right hand side of eq. (16). In this case the distribution when t tends to infinity will naturally deviate from the Maxwell-Boltzmann. It is readily inferred that j does depend not only on energy dispersion but also on the energy distribution of escaping stars: In fact King(1958*b*) considered that the star will leave the cluster as soon as its energy H becomes to zero, because the chance for a star to get more energies is very little, and also he showed that the reverse effect, namely to loose its energy from zero to negative one due to encounter while moving to cluster periphery, is negligible for general clusters (1959). Whether we adopt this consideration that an escaping star will carry out "zero" energy, or on the contrary, an assumption that all escaping stars take out energies corresponding to their velocities in the tail of Maxwellian distribution, will determine the value of j. Though the rate at which star escapes from the cluster is not same at each point, we should consider j independent on position but on time, since what is determined by encounters is distribution of total mechanical energies of stars, and velocity distribution at a point is fixed by the one at another point as shown by Woolley(1954), being the time of relaxation is much longer than the orbital period of stars in the cluster.

$$N=\iiint\limits_{|v|<v_e} f\, dudvdw \quad \ldots\ldots \quad \text{number of stars per unit volume,} \tag{17}$$

$$N\overline{\boldsymbol{v}} = \iiint\limits_{|v|<v_e} \boldsymbol{v}f\, dudvdw \quad \ldots\ldots \quad \text{flux of mass across unit area per unit time,} \tag{18}$$

$$N\Theta = \iiint\limits_{|v|<v_e} \tfrac{1}{2}|\boldsymbol{v}|^2 f\, dudvdw \quad \ldots\ldots \quad \text{kinetic energy per unit mass per unit volume, and} \tag{19}$$

$$N\,X = \iiint\limits_{|v|<v_e} \boldsymbol{v}\tfrac{1}{2}|\boldsymbol{v}|^2 f\, dudvdw \quad \ldots\ldots \quad \text{flux of energy across unit area per unit time,} \tag{20}$$

then we have from eq. (16)

$$\frac{\partial N}{\partial t} + \operatorname{div}\,(N\overline{\boldsymbol{v}}) = -\beta'\lambda N, \tag{21}$$

$$\frac{\partial N\overline{v}}{\partial t} + \mathrm{div}\,(N\overline{vv}) + N\,\mathrm{grad}\,V = -\beta'' N\overline{v}, \tag{22}$$

$$\frac{\partial N\Theta}{\partial t} + \mathrm{div}\,(N\mathbf{X}) + N\overline{v}.\ \mathrm{grad}\,V = -\beta''' \mu N\Theta, \tag{23}$$

where V is the gravitational potential of the cluster, and

$$\lambda\,(r,t) = 1-2\Phi_2\,(j(t)v_e(r,t)), \quad \mu\,(r,t) = 1-\frac{4}{3}\,\Phi_4\,(j(t)v_e(r,t)). \tag{24}$$

Eqs. (21) to (23) are analogous to the hydrodynacimal equations, but infinitely high moments of f with respect to velocity are needed to solve them completely. Eq. (21) suggest the flowing-out of stars, and envelope formation, or contration with ejection of stars. Eq. (22) shows a friction $-\beta N\overline{v}$ hindering the cluster from expanding or contracting motions accompanying structure change, and indicates that the cluster tends to equilibrium.

Let R be the "radius" of a spherically symmetrical cluster, defined as the distance at which N virtually vanishes, then the total number of stars, n, is

$$n = \int_0^R 4\pi\, Nr^2\,dr. \tag{25}$$

Then, from eq. (23) for this case, we get

$$\frac{dn}{dt} = -\int_0^R \beta\lambda N 4\pi r^2\,dr = -\overline{\beta\lambda}\,n. \tag{26}$$

Similarly, for the total kinetic energy,

$$\mathfrak{H}_K = \int_0^R 4\pi N\,\Theta\, r^2\,dr, \tag{27}$$

we have

$$\frac{d\mathfrak{H}_K}{dt} = -\int_0^R \beta\mu\, N\Theta\, 4\pi r^2\,dr = -\overline{\beta\mu}\ \mathfrak{H}_K. \tag{28}$$

5. Distribution Function of Energy and Angular Momentum.

On studying structure of stellar clusters, it seems more reasonable to consider the distribution of orbital elements of stars in the cluster, than to treat of distribution of velocity and position, especially in the case, like clusters, where the mean free paths are so long, or time of relaxation is much greater than the orbital period of a star around the cluster center. For instance, the time of relaxation of M3 is of the order of 5×10^9 years, while the periods are on the average 10^6 to 10^7 years. Therefore, stars might be supposed to continue their motion on an orbit for several revolutions, and consequently, consideration on the change of number of stars over an orbit might be more signifiicant than the variation of their number of stars in a localized small range *dudvdwdxdydz*, on the contrary to the case of a gaseous sphere in which the idea of orbit loses its meaning.

Particularly for a spherically symmetrical cluster, the orbit can be specified by its total mechanical energy, H, and angular momentum, J. After a simple manipulation the eq. (16) is written in polar coordinates:

$$\frac{\partial f}{\partial t} + v_r \frac{\partial f}{\partial r} + \left(-\frac{\partial V}{\partial r} + \frac{v_T{}^2}{r}\right)\frac{\partial f}{\partial v_r} - \frac{v_r v_T}{r}\frac{\partial f}{\partial v_T} = -\beta f + 2\pi\Phi\beta N, \quad (29)$$

where,

$$\Phi = \frac{j^3}{\pi^{\frac{3}{2}}} e^{-j^2 v^2}, \qquad v^2 = v_r{}^2 + v_T{}^2, \quad (30)$$

and v_r and v_T mean the radial and tangential components of velocity, respectively. It is, however, a question if the redistribution function takes isotropic or ellipsoidal form. All of our approximation are based on analogy of the exact theory of Brownian motion, we have adopted isotropic form, because the theory gives a Maxwellian as the ultimate distribution. This corresponds to the fact that the coefficient of dynacial friction has assumed to be isotrpic. Therefore, we might be contended with the form eq. (30) until an extended theory would have been established for the Brownian motion in a inhomogeneous assembly of particles.

As mentioned before we shall consider a case where the all member stars of the cluster have a same mass, without loss of generality putting equal to unity.

Then owing to the transformation

$$\left.\begin{aligned} v_r\ (r, H, J) &= \pm\sqrt{2H - \frac{J^2}{r^2} - 2V(r)}, \\ v_T\ (r, J) &= \frac{J}{r}\ (J \geqq 0), \\ \text{and} \qquad r &= r, \end{aligned}\right\} \quad (31)$$

with a relation

$$4\pi r^2 dr . 2\pi v_T dv_r dv_T = \frac{8\pi^2}{|v_r|} JdHdJ, \quad (32)$$

the eq. (29) takes the form in terms of H and J:

$$\frac{\partial f}{\partial t} + v_r(r, H, J)\frac{\partial f}{\partial r} = -\beta f + 2\pi\Phi\beta N. \quad (33)$$

Now we shall obtain the distribution function $\Psi\ (H, J)JdHdJ$ of energy and angular momentum in the whole cluster.

The equation for $\Psi\ (H, J)$ can be given as follows: Let us distinguish the values of f and ψ by suffices $+$ and $-$ according as the cases for $v_r > 0$ and $v_r < 0$, respectively. Then from eq. (33), we have

$$\begin{aligned} \frac{\partial \psi_+}{\partial t} + 8\pi^2 \frac{\partial f_+}{\partial r} &= -\beta\psi_+ + \frac{8\pi^2\Phi}{|v_r|}\frac{\beta N}{r^2}, \\ \frac{\partial \psi_-}{\partial t} - 8\pi^2 \frac{\partial f_-}{\partial r} &= -\beta\psi_- + \frac{8\pi^2\Phi}{|v_r|}\frac{\beta N}{r^2}, \end{aligned} \quad (34)$$

where

$$\psi_+ = \frac{8\pi^2 f_+}{|v_r|} \qquad \text{and} \qquad \psi_- = \frac{8\pi^2 f_-}{|v_r|} \ ; \ \psi = \psi_+ + \psi_-. \tag{35}$$

$\psi \, dr$ means the number of stars in a sheel of thickness dr within the range, $(H, H+dH; J, J+dJ)$. Then the total number of stars in this range is

$$\Psi \ (H, J) \ JdHdJ = \int_{p(H,J)}^{q(H,J)} \psi \ dr. \ JdHdJ), \tag{36}$$

where $p(H, J)$ and $q(H, J)$ denote the peri- and apocentric distances of the orbit specified by H and J. Thus we have

$$\frac{\partial \Psi}{\partial t} = -\int_p^q \beta \psi \, dr + \int_p^q \frac{16\pi^2}{r^2} \frac{\Phi \beta N}{|v_r|} \, dr, \tag{37}$$

here we have employed the following relation:

$$\int_p^q \frac{\partial f_+}{\partial r} dr - \int_p^q \frac{\partial f_-}{\partial r} = 0 \tag{38}$$

due to

$$f_+ (q) = f_- (q) \qquad \text{and} \qquad f_+ (p) = f_- (p). \tag{39}$$

Introducing an everage value of β :

$$< \beta > = \int_p^q \beta \, \psi \, dr \Big/ \int_p^q \psi dr = \frac{1}{\Psi} \int_p^q \beta \Psi dr, \tag{40}$$

which might mean the reciprocal of "time of relaxation of the orbit (H, J)", we have

$$\frac{\partial \Psi}{\partial t} = - < \beta > \Psi + 16\,\pi^2 \int_p^q \beta \frac{\Phi N}{|v_r|} \, dr. \tag{41}$$

The first term $-<\beta>\Psi$ means the total decrease of number of stars arising at every portion, while the second the total income from other orbits. For the study on spherical cluster we shall adopt the eq. (41) as the fundamental equation.

From the equation we can obtain the density distribution of stars in the cluster. Actually the distribution of stars on the orbit is not even; the stars are exchanged with other orbits in different rate at its each portion. However, if we approximately suppose that they are distributed uniformly on the orbit, the space density of the stars in the cluster can be computed rather easily after Eddington's method of time of stay(Eddington 1915) as

$$N(r) = \frac{1}{2\pi r^2} \iint\limits_D \frac{\Psi \ (H, J) \ JdHdJ}{T(H, J) \, |v_r(H, J, r)|} \ , \tag{42}$$

where T is the orbital period, and the integral is carried out over a range, D, for H and J, of which p and q satisfy the condition,

$$0 \leqq p < r, \ \text{and} \ r \leqq q < \infty, \tag{43}$$

respectively.

If the uniform distribution on the orbit is assumed as an approximation, there is a relation

$$\psi = \frac{2\Psi}{T|v_r|}, \tag{44}$$

hence,

$$<\beta> = \frac{2}{T}\int_p^q \frac{\beta}{|v_r|}\,dr. \tag{45}$$

Comparing to the relation $\int_p^q dr/|v_r| = \frac{1}{2}T$, it may be said that the rate of energy exchange of stars depends on a factor, the ratio of time required for stars to pass through active regions of exchange to the period of the orbit. Hence, if the active regions are concentrated towards the center, stars of every extended orbits penetrating the regions suffer from energy exchange much less than those moving only in and about the central parts. It should be noted that the stars of the orbit (H, J) interact with only those stars that exist in a shell between $p(H, J)$ and $q(H, J)$.

From eqs. (41) and (42) we can obtain an equation for timely change of N,

$$\frac{\partial N}{\partial t} = -<\overline{\beta}> N + \frac{8\pi}{r^2}\iint_D \frac{JdHdJ}{T|v_r|}\int_{p(H,J)}^{q(H,J)} p\,\frac{\Phi N}{|v_r|}\,dr \tag{46}$$

where $<\overline{\beta}>$ is the mean valve of $<\beta>$ over ohe cluster

$$<\overline{\beta}> = \iint_D >\beta> \frac{\Psi JdHdJ}{T|v_r|} \Big/ \iint_D \frac{\Psi JdHdJ}{T|v_r|} = \text{function of } r \text{ alone}. \tag{47}$$

Eq. (46) shows that the change of density at a particular point, r, is governed not only by ejection local effect of encounters at r but also by the total effects at other points, and this would be the essential point for discussing the evolution of clusters.

6. Remarks on $\overline{T}_E$.

According to Chandrasekhar(1943*b*), the coefficient of dynamical fricfriotion, η, is a decreasing function of $|v|$,

$$\eta = \eta_o\frac{2\Phi_2(y)}{y}, \qquad \eta_o = \frac{3}{\sqrt{2}}\frac{1}{T_E}, \qquad y = \frac{|v|}{\sqrt{\overline{|v|^2}}}. \tag{48}$$

Considering this fact, he evaluated λ as 0.016(1943*d*), and Spitzer and Härm (1958) as 0.0081, rather greater values than for the case $\eta=$ const. For the varying T_E after Chandrasekhar's formula(1942, eq. (2.360)), our approximation yields $\lambda = 0.001$, a much smaller value. This seems to come from the fact that f does not vanish for $|v| > v_e$ in our method and greater weights are assigned to larger values of $|v|$.

D_0 in eq. (4) is taken as the mean distance of a star to its closest neighbor following Chandrasekhar. But choice of D_0 is not definite. Cohen, Spitzer and Routly(1950) showed more distant encounters to be significant and took as D_0 the dimension of the cluster. This increases $\overline{T}_E$ by a factor of about 1.5 and affects age and life-time estimation of clusters.

7. Concluding Remarks.

It seems the fundamental problem in equilibrium theory of globular clusters to get structure of a finite mass. In so far as treating the clusters as in equilibrium, some assumptions should be made for velocity distribution, i.e., the cutoffs in H and J, as suggested by Spitzer and Härm. Only by considering the cluster as an evolving system from its initial state of a finite mass and by treating internal mechanism that governs the distribution, we could obtain the complete treatment of its structure. In non-equilibrium treatment, however, cutoffs can be automatically introduced from the condition of finite mass. In actual cases, the energy exchange is active in

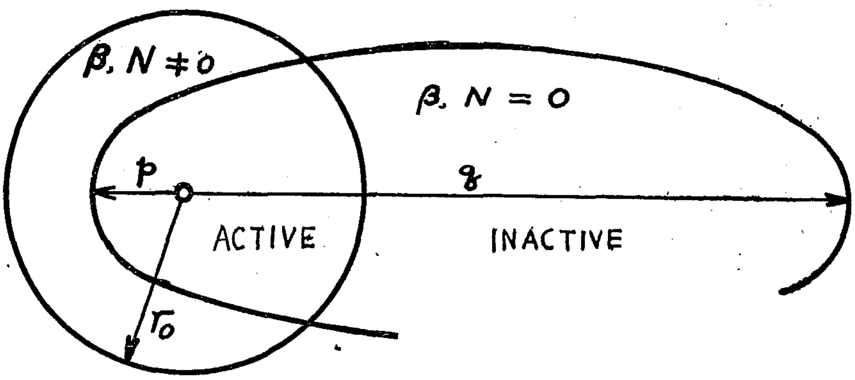

Fig. 1. An extended orbit of stars moving through the active and mactive regions of mutual exchange of energy in a cluster.

dense inner regions, and the orbits (H, J) extending to outer regions $(q > r_0)$ are produced from the stars moving through the range $p(H, J)$ and r_0, where r_0 is the radius of the active region. Eq. (41) implies that the stars on the orbit (H, J), whose peri- and apocentric distances are p and q, respectively, suffer only from stars existing between the shell of radii p and q. And it shows an significant fact: Let suppose that the energy exchange is active only in a sphere $r < r_0$, or in other words, assume that N and $\beta \neq 0$ when $r > r_0$, and N and $\beta \approx 0$ when $r > r_0$. Then, if there was no orbit of a pericentric distance p greater than r_0 at $t=0$, the equation for Ψ of such orbit becomes

$$\frac{\partial \Psi}{\partial t} = - < \beta > \Psi. \tag{49}$$

Since $\Psi = 0$ at $t=0$, it is identically zero for $t > 0$, and there should exist no orbit for which $p > r_0$ when $t > 0$. Consequently we can conclude that any orbit extending outward (i.e., $q > r_0$) must pass through the original cluster (Fig. 1). Thus the total number of stars of $q > r_0$ is finite, and the orbits have $H < 0$ and J's are within a certain limit corresponding to H and p.

Thus the stars which can move to far beyond r_0 have orbits for which $p < r_0$ and $q \gg r_0$; which means that the orbits are very much elongated. Consequently there exist preferential radial motions of stars in the outer parts of the cluster, and this fact naturally corresponds to the cutoff with respect to J. The upper limit of J is obviously $r_0 v_e^2(r_0)$, where $v_e(r_0)$ is the velocity of escape of a star at a distance r_0.

II. Timely Change of Density Distribution in a Simplified Model Globular Cluster.

8. Introduction.

At present the globular clusters are generally believed to be in equilibrium state after started from their original configuration about 5×10^9 years ago, inferred from the facts that their observed internal structures are similar to each other, and close to isothermal configuration, and secondly that energy exchange in the clusters can be regarded as to have already become effective. As stated in Chapter I, the studies on the internal constitution of globular clusters so far carried out were based on the assumption of equilibrium(Woolley 1954, von Hoerner 1957, and Oort-van Herk 1959), and the changes of cluster mass and radius with time were investigated also on the same ground (King 1958*b*). For example, M3 may certainly be considered to be very close to equilibrium configuration.

Though it is likely to be the case for general globular clusters, or more precisely for smaller ones, some of globular clusters would have to be considered to lie inbetween stages of their initial and equilibrium states. As the evidences that all clusters would not have arrived at equilibrium we may indicate following facts:—

(1) Ages of globular clusters are now estimated as about 5×10^9 to 6×10^9 years, while their times of relation are of order of 4×10^9 to 5×10^{10} years. Consequently, the clusters of longer relaxation times must leave vestiges of their original configurations, and hence the density distributions will deviate from isothermal.

(2) There exist, in fact, globular clusters, like ω Centauri, which have internal structure remarkably dissimilar to those of average clusters like M3 and M92. The dissimilarity cannot be explained by the equilibrium theory, as pointed out by Oort and van Herk(1959).

Incidentally, we might here make a remark on the equilibrium test introduced by von Hoerner(1957). As will be mentioned later, even if star counts in a globular cluster gave an approximate linear relation between density distributions of stars of different masses, we could not necessarily conclude that the cluster be in equilibrium.

The main purpose of the present chapter is to examine possibility of our hypothesis of non-equilibrium by investigating a simple model, as the first step before studying more complicated realistic cases; particularly, we shall investigate timely change of density distribution in a cluster, and as a result, show the observed distribution in ω Centauri to be well explained as still staying in non-equilibrium state.

9. Model Globular Cluster and Fundamental Equations.

For simplicity we shall consider a model cluster in which (1) all stars have a same mass, m, and (2) time of relaxation, $\overline{T_E}$, is everywhere equal througout the cluster. $\overline{T_E}$ is represented by mean relaxation time given by Chandrasekhar's formula(Chandrasekhar 1942, eq. (5.218))

$$\overline{T_E} = 8.8.10^5\sqrt{\frac{n\overline{R}^3}{\overline{m}}}\frac{1}{\log n - 0.45} \quad \text{years}, \tag{50}$$

where n is the total number of stars, $\overline{m}$ the mean mass of the stars, $\overline{R}$ the "average" radius of the cluster, in the units of solar mass and parsec, respectively.

Suppose the initial cluster be a stellar assembly limited in a finite region. Then, owing to energy exchange among stars, take place ejection of stars, formation of envelope, and change of internal structure. In these stages the cluster appears to expand, while in later stages, the configuration tends to equilibrium asymptotically with approximately contraction arising from diminution of the total mass.

The equation for timely change of density, $\rho = mN$, at a distance, r, from the center is given by eq. (46)

$$\frac{\partial\rho(r,\tau)}{\partial\tau} = -\rho(r,\tau) + \frac{16\pi^2}{r^2}\int\frac{JdHdJ}{T|v_r|}\int_p^q\frac{\Phi\rho}{|v_r|}dr, \tag{51}$$

where

$$\Phi = \frac{1}{\pi^{\frac{3}{2}}\sigma^3}e^{-\frac{v^2}{\sigma^2}(2H-2V(r))}, \tag{52}$$

$$\sigma^2 = \frac{1}{j^2}, \tag{53}$$

and

$$\tau = \int_0^t \beta\,dt. \tag{54}$$

The velocity field being assumed isotropic in the inner parts, we have from eq. (32)

$$\frac{JdHdJ}{r^2|v_r|} = v^2dv. \tag{55}$$

Hence the last term of eq. (51) becomes

$$16\sqrt{\pi}\int_0^{ve}\frac{v^2dv}{T}e^{-\frac{2H}{\sigma^2}}\int_p^q\frac{\rho ve^{\frac{2V}{\sigma^2}}}{|v_r|}dr = 16\sqrt{\pi}e^{-\frac{2V}{\sigma^2}}\int_0^{ve}\frac{v^2e^{-\frac{v^2}{\sigma^2}}}{T}dv\int_p^q\frac{e^{\frac{2V}{\sigma^2}}\rho}{|v_r|}dr, \quad (56)$$

where T, the orbital period, should be understood as a function of $H=\frac{1}{2}v^2+V(r)$.

Besides eq. (51) we must have the Poisson's equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = -4\pi G\rho. \quad (57)$$

The change of total mass of the cluster, $\mathfrak{M}$, is obtained from eq. (26) as

$$\frac{d\mathfrak{M}}{td} = -\beta\lambda\mathfrak{M}, \qquad \lambda = 1-\overline{2\Phi_2\left(\frac{v_e}{\sigma}\right)}, \quad (58)$$

where the bar means averaging over the cluster.

In order to solve these equations, the method of successive approximation is applied. As inferred from eqs. (51) and (56) with fiinite v_e, the equilibrium configuration would be something like the truncated isothermal having a finite mass and radius as shown by Woolley(1954). Thus, as the first approximation, we adopt the form

$$\rho\ (r,\tau) = \rho_0\ (\tau)\ e^{-\frac{2V}{\sigma^2}}, \quad (59)$$

where $\rho_0(\tau)$ is a function only of λ, and $V(r)$ is the potential of a truncated isothermal sphere yeilded by a dimensionless equation

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left(\eta^2\frac{d\psi}{d\eta}\right) = -V\ (\psi), \quad (60)$$

in which k means the value of $-2V(0)/\sigma^2$, and

$$\left.\begin{aligned} &V = -\frac{\sigma^2}{2}(k-\psi),\ r=l(\tau)\eta,\ \ l(\tau)=\sqrt{\frac{\sigma^2(\tau)}{8\pi G\rho'(\tau)p}},\ \ p=1-\lambda,\\ &\text{and}\\ &V(\psi)=\frac{\Phi_2(\sqrt{k-\psi})}{\Phi_2(\sqrt{k})}e^{-\psi}. \end{aligned}\right\} \quad (61)$$

Then, the equation for ρ of the second approximation becomes

$$\frac{\partial\rho}{\partial\tau} = -\rho+\rho'\ (\tau)\ pU\ (\psi), \quad (62)$$

where

$$\rho'\ (\tau)\ p = 2\rho_0(\tau)\ \Phi_2\ (\sqrt{k})\ e^k. \quad (63)$$

In order to determine σ^2, we shall adopt the King's assumption(1958*b*) that ejected stars leave the cluster with zero energy, and employ the relation derived from the virial theorem for equilibrium as the first approximation

$$\mathfrak{M}\ (\tau)\ \sigma^2\ (\tau) = \mathfrak{M}\ (0)\ \sigma^2\ (0). \quad (64)$$

The quantities $l\ (\tau)$ and $\rho'\ (\tau)$ can be determined from properties of homologously contracting truncated isothermal sphere, it is readily seen from eq. (51) that as τ tends to infiinity the cluster structure becomes that sphere specified by the quatities expressed by eqs. (61). Since the change of total mass follows eq. (58), we have

$$\mathfrak{M}(\tau) = \int_0^{R(\tau)} 4\pi r^2 \rho dr = 4\pi\rho'(\tau)\, l(\tau)^3\, q(\tau)\, p = \mathfrak{M}(0)\, e^{-\lambda\tau}, \tag{65}$$

$$l(\tau) = l(0)\, e^{-2\lambda\tau}, \qquad \rho'(\tau) = \rho'(0)\, e^{5\lambda\tau}, \tag{66}$$

$$l(0) = \frac{2G\mathfrak{M}(0)}{\sigma^2(0)} \frac{1}{q}, \qquad \text{and} \qquad \rho'(0) = \frac{\sigma(0)^2}{8\pi G p l(0)^2}, \tag{67}$$

where q is the reduced mass of truncated isothermal sphere, expressed as

$$q(\tau) = \left(-\eta^2 \frac{d\psi}{d\eta}\right)_{R(\tau)/l(\tau)} \qquad \text{for } \psi\left(\frac{R(\tau)}{l(\tau)}\right) = k, \quad \eta = \frac{R(\tau)}{l(\tau)}, \tag{68}$$

where $R(\tau)$ is the radius of this sphere, and k is assumed independent on t.

For the initial cluster, a polytropic sphere of index n' will be adopted to simplify calculus. On putting its mass and radius $\mathfrak{M}(0)$ and R_0, respectively, then we get relations

$$\left.\begin{aligned} \sigma(0)' &= \frac{2}{5-n'} \frac{G\mathfrak{M}(0)}{R_0}, \\ l(0) &= (5-n') \frac{R_0}{q}, \\ \rho'(0) &= \overline{\rho(0)} \frac{q^2}{3(5-n')^3 p}, \qquad \overline{\rho(0)} = \frac{\mathfrak{M}(0)}{\frac{4\pi}{3} R_0^3} \end{aligned}\right\} \tag{69}$$

because the total kinetic energy of the cluster is half of the absolute value of its potential energy which is equal to $\frac{3}{5-n'} \frac{G\mathfrak{M}(0)^2}{R_0}$ (Eddington 1926). Expressing $\rho(r,\tau)$ by $\theta(z,\tau)$ in the unit of central density $\rho n'(0,0)$ of the initial cluster, and r by the polytropic dimensionless distance, z; namely by putting

$$\theta(z,\tau) = \frac{\rho(r,\tau)}{\rho_{n'}(0,0)}, \qquad z = \frac{z_{n'}}{R_0} r, \qquad z_{n'} = \text{polytropic dimensionless raeius,} \tag{70}$$

then eq. (62) becomes

$$\frac{\partial\theta(z,\tau)}{\partial\tau} = -\theta(z,\tau) + p g_{n}' e^{5\lambda\tau}\, V\left[\psi(h_n' e^{2\lambda\tau} z)\right], \tag{71}$$

where

$$g_{n'} = \frac{q^2}{3(5-n')^3 C_{n'} p}, \quad h_{n'} = \frac{q}{(5-n') z_{n'}}, \quad \text{and } C_{n'} = \left(-\frac{z}{3}\frac{dz}{du}\right)_{u_{n'}=0}. \tag{72}$$

10. Numerical Results.

For the initial cluster, we shall consider two cases of polytropic indices $n' = 1.5$ and 3, the former having lower and the latter higher degree of mass concentration, respectively.

TABLE 1. VALUES OF -LOG θ (z, τ) FOR THE CASE, $n' = 1.5$.

log τ / τ / log z	-1.0 0.100	-0.5 0.316	0.0 1.00	0.5 3.16	1.0 10.0	1.5 31.6
$-\infty$	0.008	0.003	−0.013	−0.045	−0.155	−0.502
−1.2	.009	.005	− .004	− .034	− .140	− .470
−1.1	.013	.010	.003	− .025	− .129	− .448
−1.0	.017	.014	.012	− .013	− .110	− .420
−0.9	.021	.021	.027	.009	− .085	− .379
−0.8	.022	.028	.046	.037	− .049	− .313
−0.7	.027	.040	.071	.081	.005	− .218
−0.6	.032	.054	.107	.147	.090	− .095
−0.5	.038	.074	.157	.243	.207	.060
−0.4	.047	.096	.219	.359	.350	.250
−0.3	.061	.122	.285	.506	.527	.471
−0.2	.080	.051	.351	.673	.739	.725
−0.1	.110	.189	.420	0.860	0.980	0.977
0.0	.151	.236	.490	1.046	1.232	1.242
0.1	.217	.305	.574	1.219	1.485	1.494
0.2	.324	.413	.690	1.398	1.742	1.749
0.3	.499	.589	0.870	1.608	1.979	2.000
0.4	0.798	0.888	1.166	1.890	2.257	2.259
0.5	1.428	1.513	1.768	2.323	2.514	2.536
0.6	3.851	3.331	2.950	2.776	2.768	2.840
0.7	4.078	3.606	3.228	3.055	3.070	3.170
0.8	4.308	3.902	3.532	3.361	3.388	3.639
0.9	4.587	4.211	3.866	3.691	3.760	4.300
1.0	5.053	4.680	4.398	4.251	4.550	—

TABLE 2. VALUES OF -LOG θ (z,τ) FOR THE CACE, $n' = 3$.

log τ / τ / log z	−1.0 0.100	−0.5 0.316	0.0 1.00	0.5 3.16	1.0 10.0	1.5 31.6
$-\infty$	0.016	0.048	0·118	0.169	0.070	−0.277
−1.2	.018	.050	.124	.177	.081	− .243
−1.1	.022	.054	.128	.182	.090	− .221
−1.0	.024	.058	.138	.195	.108	− .194
−0.9	.028	.063	.145	.214	.130	−.149
−0.8	.030	.068	.160	.340	.162	−.087
−0.7	.034	.077	.178	.275	.208	.007
−0.6	.045	.092	.208	.332	.282	.131
−0.5	.053	.107	.248	.418	.386	.287
−0.4	.068	.131	.297	.522	.520	.473
−0.3	.088	.157	.348	.654	.686	.700
−0.2	.125	.210	.422	.809	0.887	0.950
−0.1	.170	.252	.495	0.983	1.120	1.202
0.0	.245	.331	.590	1.172	1.378	1.460
0.1	.356	.444	.713	1.359	1.622	1.720
0.2	.522	.606	0.881	1.561	1.880	1.970
0.3	0.745	0.835	1.110	1.812	2.140	2.225
0.4	1.048	1.137	1.410	2.093	2.307	2.480
0.5	1.469	1.556	1.819	2.423	2.650	2.760
0.6	2.074	2.152	2.379	2.797	2.905	3.068
0.7	2.882	3.320	3.026	3.154	3.193	3.390
0.8	4.410	3.988	3.658	3.488	3.503	3.900
0.9	4.717	4.298	3.965	3.810	3.838	—
1.0	5.065	4.674	4.387	4.262	4.470	—

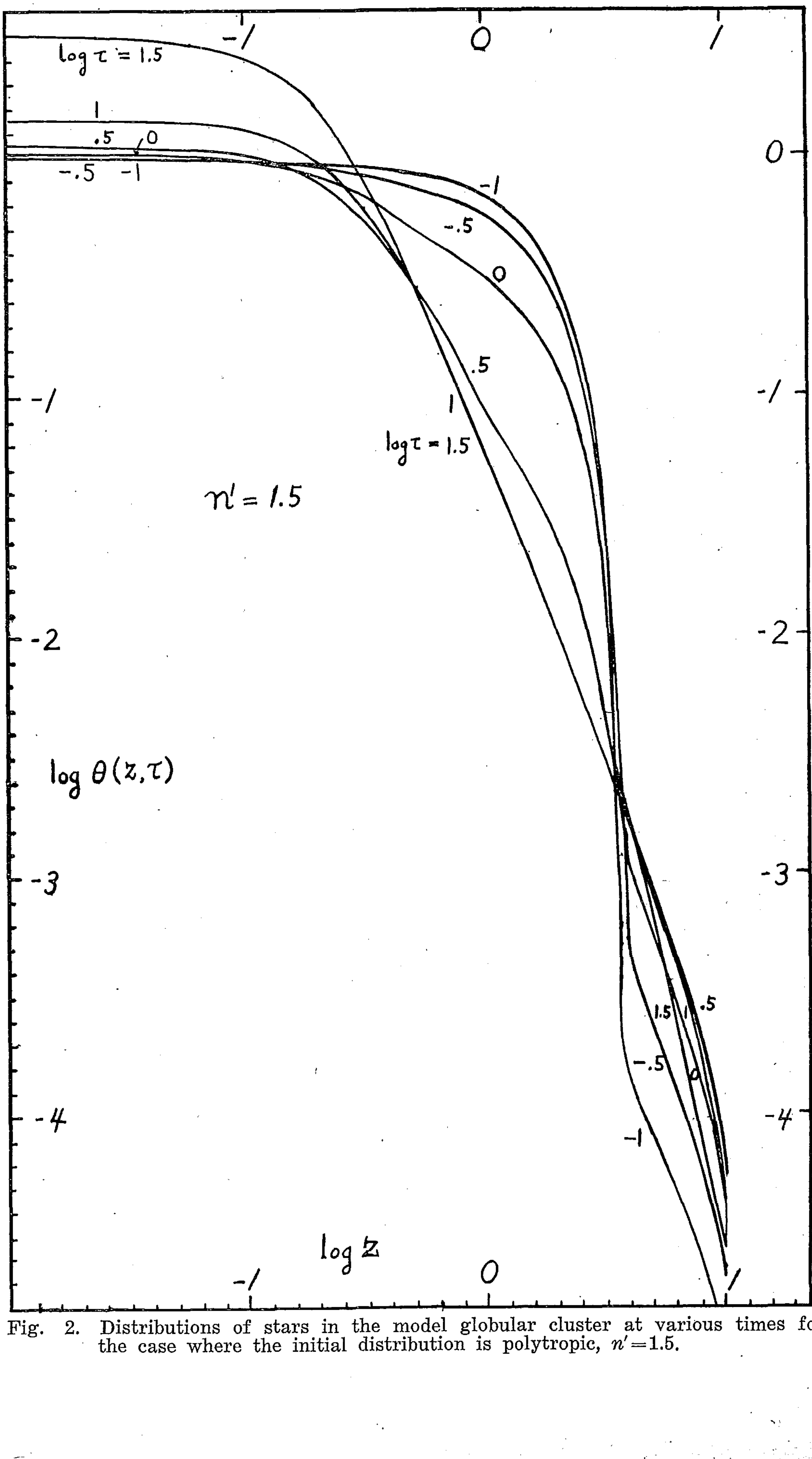

Fig. 2. Distributions of stars in the model globular cluster at various times for the case where the initial distribution is polytropic, $n'=1.5$.

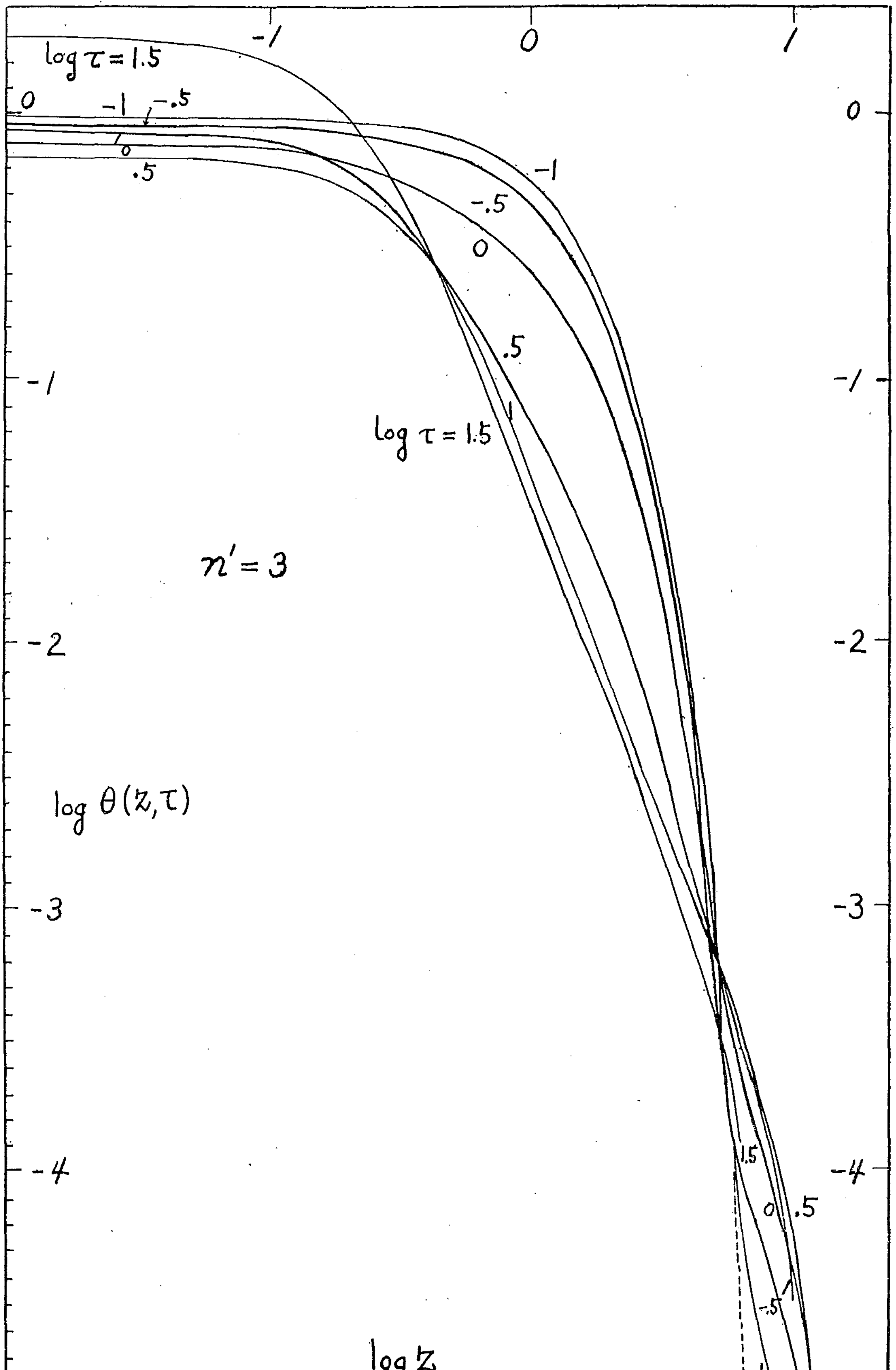

Fig. 3. Distributions of stars in the model globular cluster at various times for the case where the initial distribution is polytropic, $n'=3$.

The Emden functions of the spheres were taken from the tables by Sadler and Miller(1932).

The values of $\overline{\Phi_2(x_e)}$ is computed with the value of v_e/σ equal to$\sqrt{6}$ (Chandrasekhar 1942, p. 207), giving $\lambda=0.00738$.

Numerical solution for the truncated isothermal sphere is taken from the Woolley's Table 1(1954) for $k=7.82$.

After evaluating θ for the second approximation, we examined its accuracy by promoting to the next approximation, but found the error in the second approximation to be about 0.5% as a mean and 2% even in the worst cases. Thus the solution by eq. (71) is satisfactory accurate for out present problem.

The values of -log $\theta(z,\tau)$ from eq. (71) for log $\tau=-1.0$, -0.5, 0.0, 0.5, 1.0 and 1.5 are indicated in Tables 1 and 2 against log z when $n'=1.5$ and 3, respectively.

The Figs. 2 and 3 show the distributions of stars at each time for the both cases of initial cluster, respectively. It is seen that the cluster remains initial character until about log $\tau=-1$, and then the shoulder of the curve is scraped away to become more gentle with growing envelope. After log $\tau=1.0$, the distribution is virtually truncated isothermal.

The central density $\theta(\mathrm{o},\tau)$ decreases at the very early stages but increases again. The decrease comes from the flowing-out of stars of the central regions to form envelope, which may be here defined tentatively as the regions: $r>R_0$. On the other hand, the later increases is due to escape of stars of zero energy, namely due to decreases of mutual distances of stars following mass diminution with invariant total energy of the cluster. Fig. 4 indicates the change of $\theta(\mathrm{o},\tau)$ with time.

Let us define the "degree of compactness", κ, as a ratio of radii R^* and $R_{0.1}$ where R^* and $R_{0.1}$ mean the radii at which the densities become 10^{-4} and 10^{-1} of the central one, respectively. Then, if κ is much greater than unity, the cluster may seem to be compact, and vice versa. As shown in Fig. 5, the older the cluster, it becomes more compact.

11. Comparison with Observations for Density Distribution.

Since the actual globular clusters consist of stars of various masses and observations of internal structure are rather inaccurate, it seems difficult to compare the results from the model with the observations. However, it is not meaningless to examine if our concept be valid, and it might be possible to infer the general character of evolution of globular clusters.

Observations of internal distribution are collected for eight clusters: M2, M3, M5, M13, M15, M22, M92 and ω Centauri. We adopt the data by Hogg(1932) for M2, M13, M15, and M22, and those by Lohmann(1936) for M5, M15 and M92. They investigated photometrically for inner regions and by star count for outer parts. Therefore, their results will rather represent

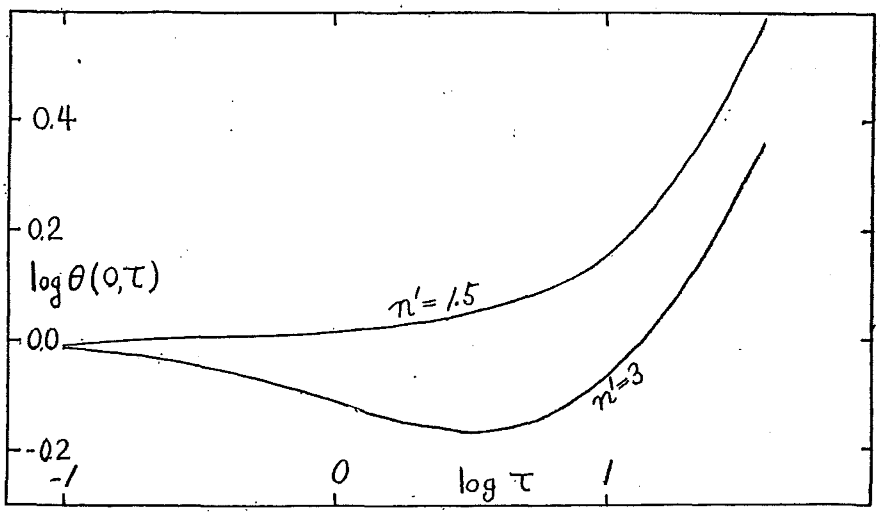

Fig. 4. Changes of the central density of the model globular cluster for the cases, $n'=1.5$ and $n'=3$.

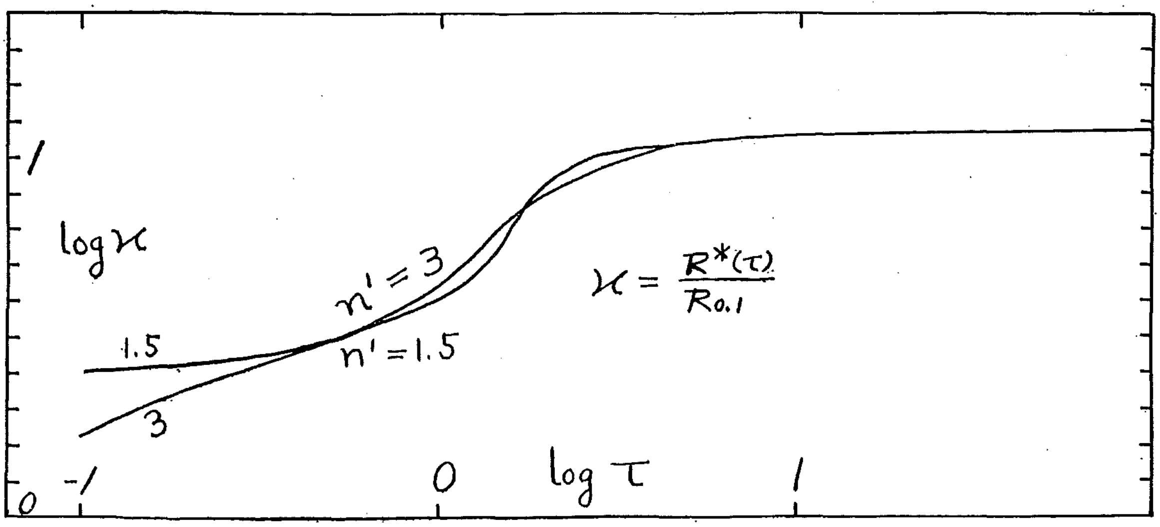

Fig. 5. Changes of the degree of compactness of the model globular cluster for the cases, $n'=1.5$ and $n'=3$.

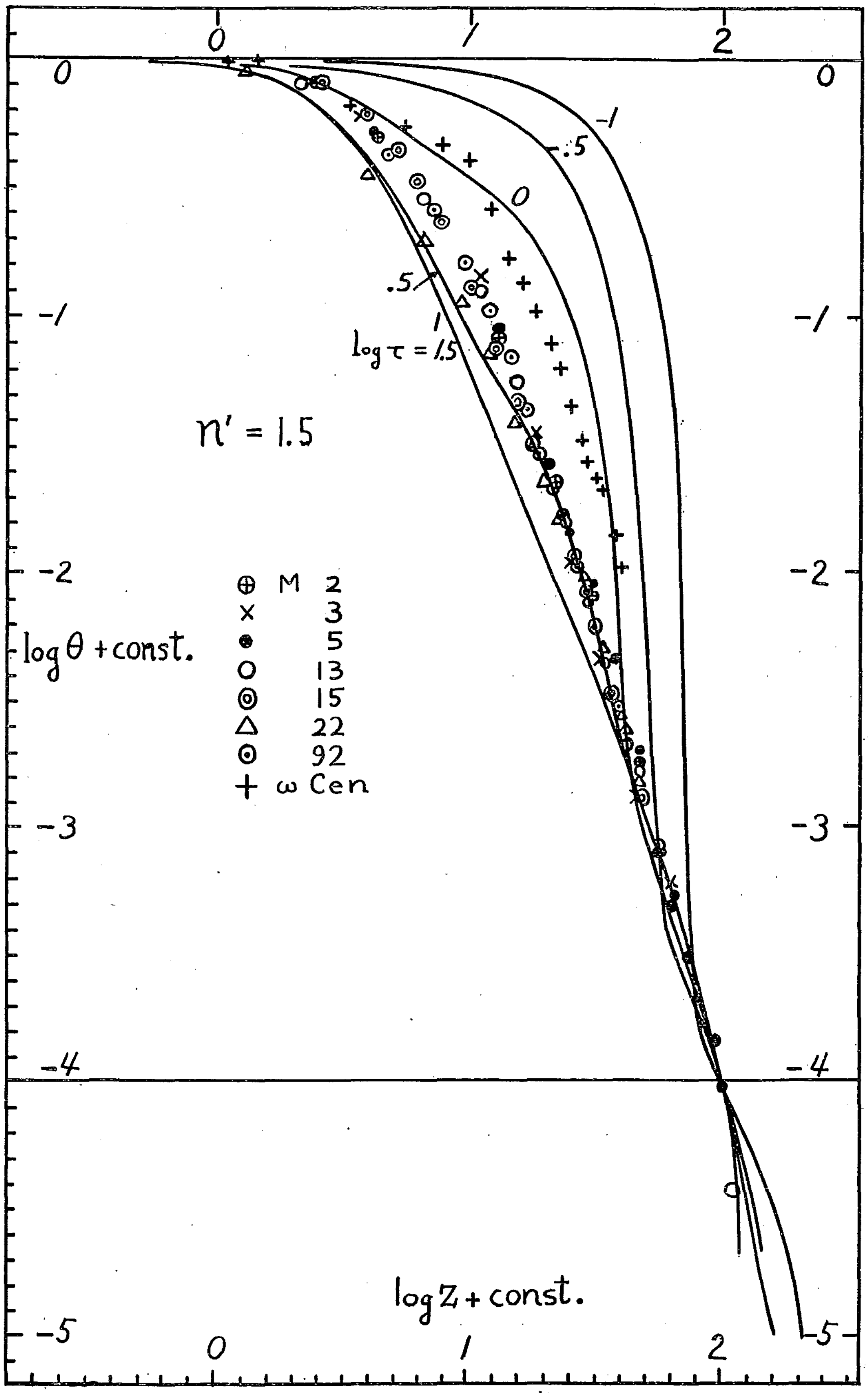

Fig. 6. Comparison of the computed density distribution at each time with the observed ones for the case, $n'=1.5$.

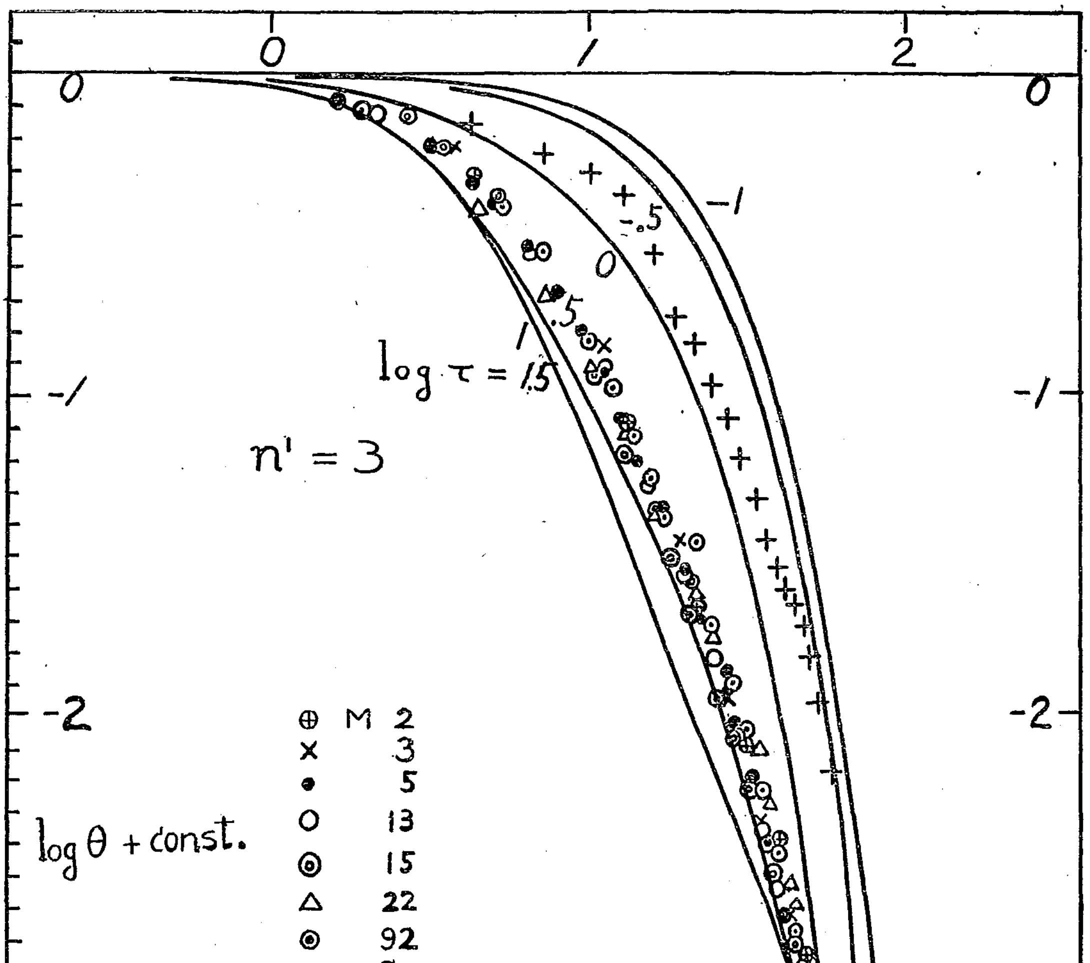

Fig. 7. Comparison of the computed density distribution at each time with the observed ones for the case, $n'=3$.

distributions of relatively brighter stars. For M3 we shall adopt the result by Hertzsprung(1918); star counts being made by Sandage(1954) for M3, and by Tayler(1954) for M92. Comparison for M15 does not show appreciate discrepancy between the observations by Hogg and Lohmann. For ω Centauri, the Schilt's result(1918) is utilized.

In order to see the characteristic differences in distributions at various stages, it is necessary to compare the shapes of curves in Figs. 2 and 3 by eliminating scale factors. The tops and points of the curves at which $\theta(z, \tau)$ becomes 10^{-4} of $\theta(0, \tau)$ were put together, respectively, as shown in Figs. 6 and 7. The number attached to each curve indicates the value of log τ. Then, the observed distribution in a cluster is put on the Figures so as to fit at the top and the 10^{-4} point, or so as to fit to a curve with some value of log τ interpolated by inspection. As seen from the Figures, the case where the initial cluster is a polytropic sphere $n'=3$ seems to give better results. It is obvious that ω Centauri fits very well to a curve with a smaller τ. The other clusters lie between log $\tau=0$ and log $\tau=0.5$, fitting fairly well. But these may have rather values larger of τ than estimated by fitting, because they might represent relatively brighter stars, which are more concentrated towards center and would give steeper gradient than distribution of stars of mean mass with the same large τ. Even if this is true, however, we can see characteristic differences between these clusters, as they must be considered to have almost a same mass function.

12. Another Verification for Structure Change.

We shall now attempt to verify by another method for the clusters to change their structure in the way discussed above. Let us denote by D the distance from the curve log $\tau=1.5$ to an observed curve at the level log $\theta+$ const.$=-0.4$ in Fig. 7 of the case $n'=3$. Then, there should be a relation between $\overline{T_E}$ and D, if the clusters had been formed at a same time. For the values of $\overline{T_E}$ we may apply their present values with sufficient accuracy. On computing $\overline{T_E}$ by means of the formula (50), n was taken as the first approximation from the determination by Matsunami et. al.(1959), based on Mowbray's diameters(1946). In our case, however, the radii listed in Becvar's catalog(1959) were adopted, and by considering von Hoerner's estimation of the radius of the active region of energy exchange in M3, we adopt halves of the listed radii as the data for the "average" ones. The values of $\overline{T_E}$ so evaluated are given in Table 3.

In Fig. 8, D is plotted against T_E. The curves indicated as 10^9, 5×10^9 and 10^{10} are the theoretical relations obtained from the solution, and mean that the clusters should be sitiated on either of them, if the clusters had been formed at the same time 10^9, 5×10^9 or 10^{10} years ago, respectively. The fitting of the clusters on the curve 5×10^9 years is considerably good, and scattering of points will be attributed to errors in observations and in approximations employed so far.

Thus, it may be said roughly that the clusters were formed about same time, approximately 5×10^9 years ago, and it is seen form $\tau \sim t/\overline{T}_E$ that the structure on smaller cluster is close to truncated isothermal and looks like older than larger cluster.

Incidentally, we shall compare the observed classes of concentration by Shapley and Sawyer (1927) with the values of $\log \kappa$ of the clusters obtained by means of Fig. 5 from $\log \tau$. listed later. Fig. 9 shows the result, and it is seen that the clusters will move to class of higher concentration as they evolve. The Figure is for the case $n'=3$.

TABLE 3. TOTAL NUMBERS, "AVERAGE" RADII, AND TIMES OF RELAXATION OF THE EIGHT GLOBULAR CLUSTERS.

Cluster	n	$\overline{R}$ (pc)	$\log \overline{T}_E$ (yr)
M2	5.4×10^5	8.0	9.66
M3	3.2	8.8	9.62
M5	3.4	10	9.70
M13	2.6	7.5	9.48
M15	3.4	6.7	9.45
M22	1.9	8.5	9.51
M92	2.0	6.7	9.35
ω Cen	8.5	12	9.98

13. Estimation of Ages of the Clusters.

The position and shape of an observed density curve can yield the age of the cluster by reading $\log \tau$ in Fig. 7. Though time of relaxation is a function of time, we can prove that τ can be expressed with the present value of $\overline{T}_E$ as

$$\tau \sim \frac{t}{\overline{T}_E}. \tag{73}$$

This formula is sufficiently accurate for our present problem. Another way is to read the age on Fig. 8 from the horizontal distance of the point from either curve, by assuming that the scattering of the points were attributed to difference in ages. Though the two ways should yield same results, this is not necessarily the case owing to reading errors. Taking both results into consideration, we have estimated the ages of the clusters as shown in the fourth column of Table 4. The Table also indicates $\log \tau$ and D obtained from Fig. 7, The ages so estimated can be compared with those derived from turn-off points on Hertzsprung-Russell diagrams. Sandage estimated the age of M3 as 5.1×10^9 years(1957), Hoyle and Schwarzschild gave 6×10^9 years (1955), and by Hoyle and Hazelgrove(1956) it is 6.5×10^9 years. The agreement of our value is good. For M2, M5 and M13, Arp(1959) assigned the age 20×10^9 years to M5 from the calculation of evolution by Hazelgrove and Hoyle, and showed that it was the oldest of the three, and M2 the intermediate, and M13, the youngest. Our estimation gives about half for M5, but the sequence of ages of the three clusters agrees with that by Arp. The age of M22 seems to be estimated rather greater, but there is no evidence to verify it. In any way, our estimation is likely to give reasonable results. The ages from the theory of stellar evolution are given in the fifth column of Table 4.

TABLE 4. LOG τ, D AND ESTIMATED AGES OF THE EIGHT GLOBULAR CLUSTERS.

Cluster	log τ	D	Age in years	Age (yrs.) from H. R. diagram and its reference
M2	+0.29	0.12	8.1×10^9	——
M3	.22	.17	6.5	6.5×10^9 Hoyle and Hazelgrove(1956).
M5	.34	.09	11	20 Arp(1959).
M13	.34	.09	6.6	7.6 Baum et. al.(1959)
M15	.34	.09	6.6	——
M22	.50	.02	9.3	——
M92	+0.33	.10	4.6	——
ω Cen	−0.29	0.55	5.1	——

As the sources of discrepancies above mentioned will be due to the facts: (i) inexactness of our model calculation, (ii) errors in observed luminosity distribution or star counts, and (iii) ambiguity of $\overline{R}$ on evaluating $\overline{T}_E$. For (i) we can estimate the exact values of log τ as larger than those given in Table 4 by 0.1 to 0.3, because the observed distribution represents that of brighter stars as mentioned before, and these corrections would make the ages by 1.3 to 2 times longer. (ii) The probable errors of the observed material now adopted might be on the average about $\pm 0.^{m}1$, giving spreads of points in Figs. 6 and 7 by about $\pm$ 0.04 in log θ, which, however, is within the error of the present model calculation. (iii) The ambiguity in $\overline{R}$ will be most effective to $\overline{T}_E$. The choice of $\overline{R}$ is rather arbitrary. King(1958*a*) adopted the radius r_0 including half the mass of the cluster in projection. According to von Hoerner's estimation(1957) of $\delta E/E$ for M3, it becomes 100% at $\frac{1}{3}R$, and 60% at r_0, which is about $\frac{1}{1.5}R$. Thus we used $\frac{1}{2}R$ as a

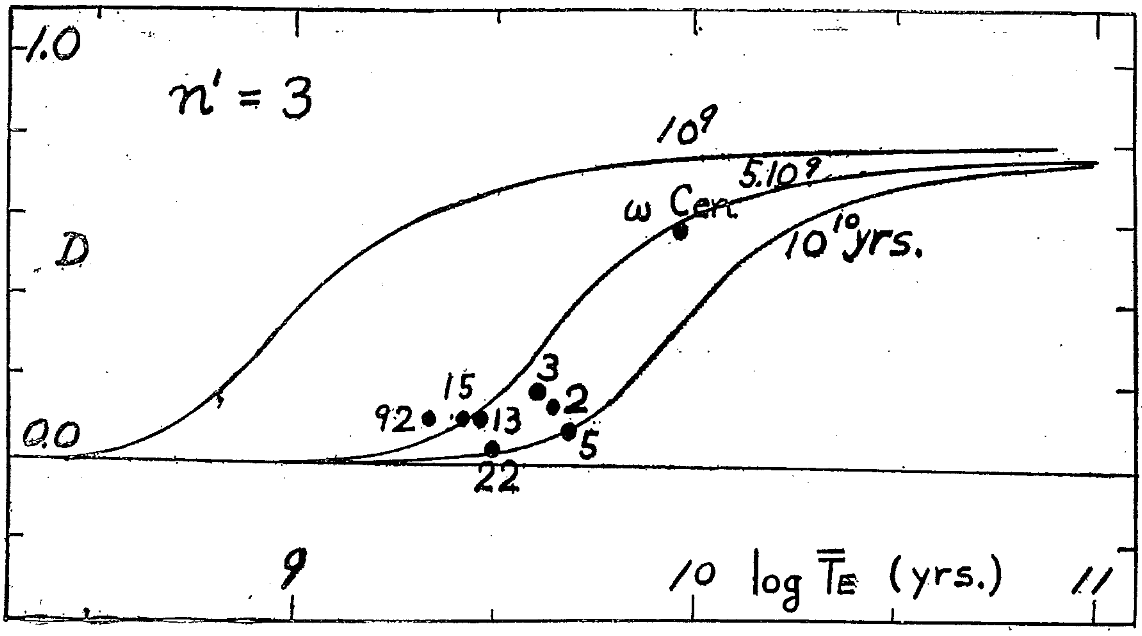

Fig. 8. Relation between the time of relaxation and deviation from equilibrium for globular clusters evolving from the initial cluster with $n'=3$. Dots denote observed clusters and curves indicate the computed relations one of which the clusters must lie if they have an same age denoted by figures attached. The Messier numbers and name of the clusters are given beside the dots.

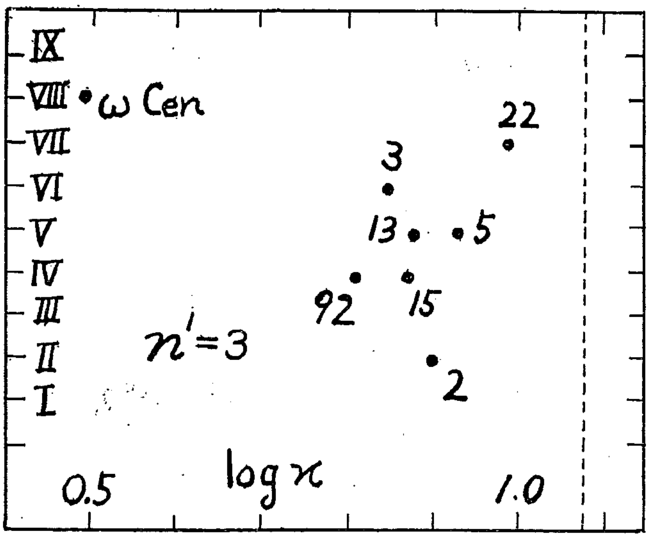

Fig. 9. Relation between the Shapley's observed concentration class and the degree of compactness for the case, $n'=3$. The Messier number and name of the clusters are given beside the dots.

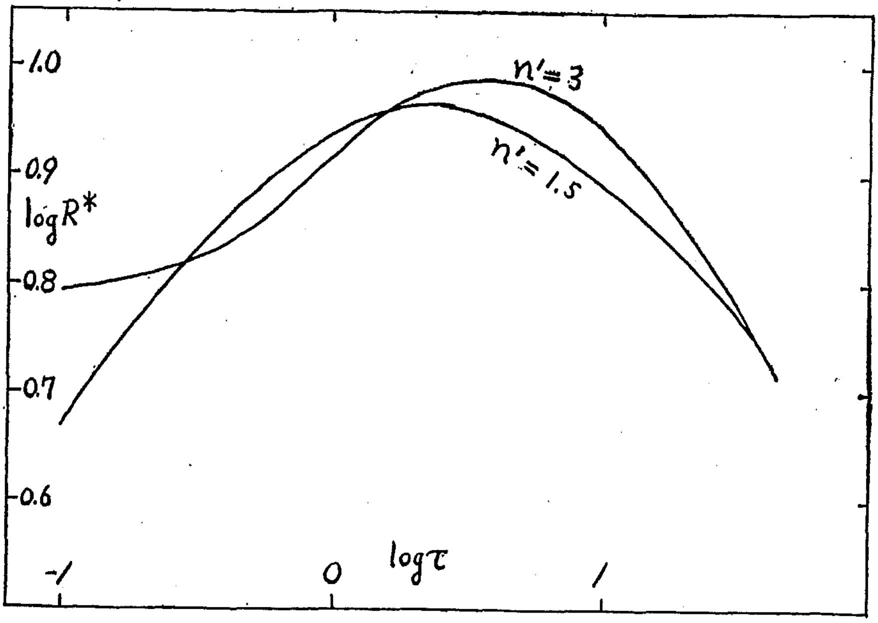

Fig. 10. Change of radius R^* for the cases, $n'=1.5$ and $n'=3$. The unit of R^* is $R_0/z_{n'}$.

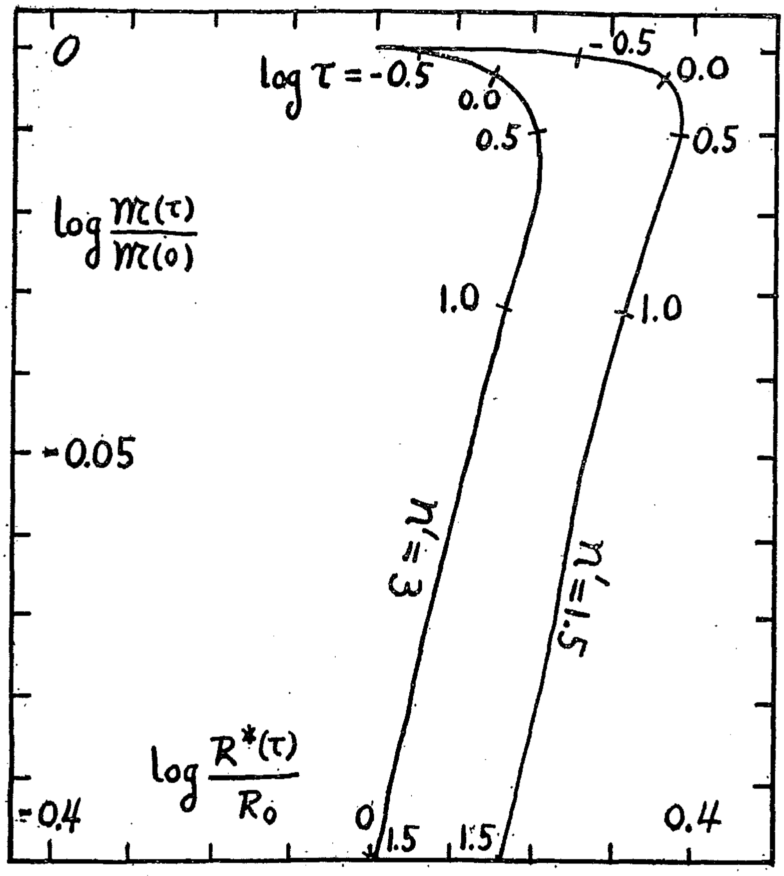

Fig. 11. Mass-radius relation for globular clusters for the cases, $n' = 1.5$ and $n' = 3$.

reasonable empirical choice.

It should be remarked that $\overline{T_E}$ has been computed for the present values of n and $\overline{R}$. The total number, n, does not change appreciably from $n(0)$ of the initial cluster (see eq. (58)). The radius R, on the other hand, has increased by about 1.5 times the initial radius as shown in Fig. 10 except for ω Centauri. In this case, eq. (50) will also be applicable as a good approximation, though based on the assumption of statistically stationary state. The factor makes $\overline{T_E}$ longer by about 1.8 times of $\overline{T_E}(0)$. Therefore, the variation of R during evolution would possibly affect the estimated ages.

14. Change of Radius and Mass-Radius Relation.

The definition of observed radius of a cluster is rather ambiguous. Though the truncated isothermal sphere has a finite radius, it is not be able to be identified with observation. Examining the observed radii, e.g., Mowred and blue limiting diameters(1946) and Becvar's listed ones(1959), there seems no defiinite relation to exist between the radii and densities at these distance from the center. But it is likely for the observed radius to indicate a distance of densities 10^{-4} to 10^{-5} time the central density. Thus, we may tentatively define the "observed" radius to be a distance R^* at which the density becomes 10^{-4} of the central one.

Owing to the formation of envelope from the central regions, R^* appears to increase at the early non-equilibrium stages, and decreases as the cluster approaches to equilibrium state, in which it contracts homologously as a truncated isothermal sphere. The timely change of R^* in the unit of $R_0/z_{n'}$ is shown in Fig. 10, for the both cases of initial cluster, $n'=1.5$ and 3.

The mass-radius relation is given in Fig. 11. On the contrary to the case of radius, the change of total mass by eq. (58) is monotonous and is approximately same as that derived from the equilibrium theory. The short bars across the tracks indicate the positions at various times in terms of τ. Referring to Table 4, we can see that the clusters now concerned would be still in the expanding stages.

15. Remark on an Equilibrium Test.

As mentioned in Sec. 8, a test for equilibrium was introduced by von Hoerner, i.e., to examine linearity of relation between counts of two kinds of stars having masses m_1 and m_2, respectively. Though this is quite an elaborate method, it seems rather difficults to apply to practical cases. Illustratively, estimates are made for this relationship: Let us consider two casks of mass pairs, (i) $m_1=\overline{m}$, $m_2=\frac{1}{2}\overline{m}$, and (ii) $m_1=\overline{m}$, $m_2=1\frac{1}{2}\overline{m}$, whese $\overline{m}$ is the mean mass of the cluster. On computing density for each kind of stars, we employ approximate form of solution of eq. (61), being variation of β with mass after Chandrasekhar's formula(1942, eq. (5.405)) and assumed relation $\sigma^2 \propto m^{-1}$ taken into consideration.

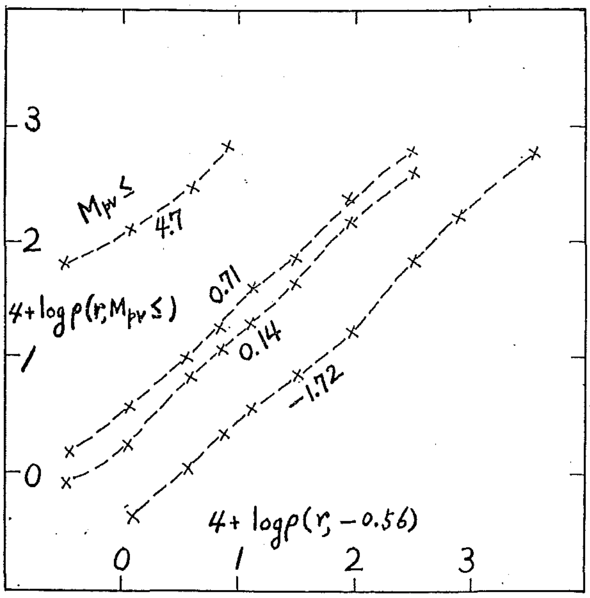

Fig. 12. Equilibrium test (after S. von Hoerner, 1957, Ap. J., **125**, 451, Fig. 7.).

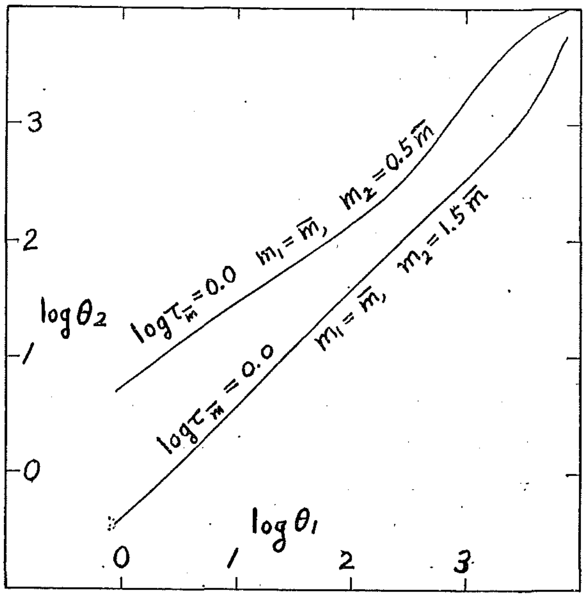

Fig. 13. Computed relations between the distributions of stars of masses m_1 and m_2.

The computed relation for log τ =0.0 is shown for each mass pair in Fig. 13 in arbitrary unit. Comparing with observed results for M3 taken from von Hoerner's paper(1957, Fig. 7), that is given in Fig. 12, we can see that the trends of curves in both Figures are very similar; the O–C seems to be of the order of magnitude within the observational errors. Thus it may be said doubtful to conclude that a cluster would be in equilibrium, even if the observed counts indicate appropriate linearity, and that observation would require considerable accuracy for this purpose that should, however, be very difficult.

16. Concluding Remarks.

So far we have considered a model cluster of stars of equal mass and of uniform rate of energy exchange, but naturally this is not the actual case. The difference in masses of the stars affects on the potential and makes each kind of stars distribute in different way from each other. The more massive the stars, they are concentrated towards the central parts, and also the rate of energy exchange is the greater. Thus the outer parts consist of less massive stars and would remain the property of the initial cluster comparatively well, much more stars of elongated orbits suffer less from other stars as noticed in Sec. 5. Therefore, it would be more effective to investigate the features in the outer parts in order to study evolution stages and properties in the past.

It is a question whether the initial clusters were limited in a finite region and all the stars in the present envelope have been evaporated from the central parts. But the present results in this chapter seems to support the hypothesis, especially as inferred from the agreement of ω Centauri with the computation based on this hypothesis. Though we cannont, of course, conclude that the initial cluster would be a polytrope 3 rather than 1.5, we may consider the initial cluster was of as much mass concentration as the polytrope 3.

Conclusions.

(1) The structure of globular clusters can be completely understood only by considering it as changing after a sequence of their evolutionary history. The dynamical theory of the globular clusters should be built up on the theory of Brownian motions for the motions of stars, and an approximate method is considered. As the conclusions: (i) resultant fundamental equation indicates that any particular restriction on velocity distribution including cutoff for angular momentum is not necessary, but can be introduced automatically to get a finite mass model, whereas in equilibrium theory various restrictions, which are usually artificial, are needed. (ii) The cluster which started from its initial state tends asymptotically to equilibrium. The rate of escape of stars from a spherical cluster can be expressed as eq. (26) in a similar form to that of an inifinitely extending homogeneous cluster. (iii) There exist preferential radial motions of stars in the outer parts of the cluster.
(2) The non-equilibrium hypothesis is also required from the point of view stated in Sec. 8. By investigating a model globular cluster, it is shown that the computed structure agrees well with observations, and it is concluded: (i) A cluster will tend at the first stage comparatively quickly to a truncated isothermal structure that has a finite mass and radius, and then changes homologously. (ii) Some of the globular clusters, e.g., ω Centauri are still in the first stage, and its peculiar internal distribution of stars can be explained only by non-equilibrium hypothesis. The structural differences between clusters might be explained in this way. (iii) At the first stage the cluster expands and then homologously contracts. (iv) General clusters are comparatively close to equilibrium. (v) The clusters are generally still expanding stages. (vi) The ages estimated from the characteristics of internal distribution generally coincide with those from the theory of stellar evolution, and this will be available to estimate the ages of the clusters.

Acknowledgements.

The author wishes to express his sincere gratitude to Professor M.

Kaburaki of the University of Tokyo for a number of valuable suggestions and encouragement. The author is also indebted to Drs. W. Unno and S. Aoki for their helpful discussions on this problem.

References.

Agekyan, T. A. 1959*a*, *A. J. U. S. S. R.*, **36**, (3, 46 in American ed.).

——— ,1959*b*, *A. J. U. S. S. R.*, **36** (3, 280 in America ed.).

Ambartsumyan, V. A. 1938, *Ann. Leningrad State Univ.*, No. 22, 19.

Arp, H. 1959, *A. J.*, **64**, 441.

Baum, W. A., Hiltner, W. A., Johnson, H. L. and Sandage, A. R. 1959, *Ap. J.* **130**, 749.

Becvar, A. 1959, *Atlas Coeli* II, *Katalog* 1950.0, 289 (Praha: Nakladatelstvi Ceskoslovenske Akademie ved Praha).

Chandrasekhar, S. 1942, *Principles of Stellar Dynamics* (Chicago: University of Chicago Press).

——— . 1943*a*, *Rev. Mod. Phys.*, **15**, 1.

——— . 1943*b*, *Ap. J.*, **97**, 225.

——— . 1943*c*, *Ap. J.*, **97**, 263.

——— . 1943*d*, *Ap. J.*, **98**, 54.

Cohen, R. S., Spitzer, Jr., L., and Routly, P. McR. 1950, *Phys. Rev.* **80**, 280.

Eddington, A. S. 1915, *M. N.*, **75**, 366.

——— . 1926, *Internal Constitution of the Stars* (Cambridge: Cambridge University Press).

Hertzsprung, E. 1918, *A. N.*, **207**, 89.

Hoerner, S. von. 1957, *Ap. J.*, **125**, 451.

Hogg, F. S. 1932, *A. J.*, **42**, 77.

Hoyle, F. and Schwarzschild, M. 1955, *Ap. J.*, *Suppl.*, **2**, No. 13.

Hoyle, F. and Hazelgrove, C. B. 1956, *M. N.*, **116**, 527.

Jeans, J. H. 1925, *Dynamical Theory of Gases*. 4th ed. (Cambridge: Cambridge University Press.)

King, I. 1958*a*, *A. J.*, **63**, 109.

——— . 1958*b*, *A. J.*, **63**, 114.

——— . 1959, *A. J.*, **64**, 351.

——— . 1960, *A. J.*, **65**, 122.

Kurth, R. 1957, *Introduction to the Mechanics of Stellar Systems* (London, New York and Paris: Pergamon Press).

Lohmann, W. 1936, *Zs. f. Ap.*, **12**, 1.

Matsunami, N., Obi, S., Shimoda, M., Takase, B. and Takebe, H. 1959, *Publ. Astr. Soc. Japan*, **11**, 9.

Mowbray, A. G. 1946, *Ap. J.*, **104**, 47.

Oort, J. H. and Herk, G. van. 1959, *B. A. N.*, 14, 299.
Owaki, N. 1960, *Contr. Mar. Res. Lab., J. H. O.*, 2, 51.
Plummer, H. C. 1911, *M. N.*, 71, 462.
Rosenbluth, M. N., MacDonald, W. M. and Judd, D. L. 1957, *Phys. Rev.*, **107**, 1.
Sadler, D. H. and Miller, J. C. P. 1932, *British Association Tables*, vol. 2 (London: British Association).
Sandage, A. R. 1954, *A. J.*, 59, 162.
—— . 1957, *Ap. J.*, **126**, 326.
Schilt, J. 1928, *A. J.*, 38, 109.
Shapley, H. and Sawyer, H. B. 1927, *Harvard Bull.*, No. 849.
Spitzer Jr., L. 1940, *M. N.*, **100**, 396.
Spitzer Jr., L. and Härm, R. 1958, *Ap. J.*, **127**, 544.
Tayler, R. J. 1954, *A. J.*, 59, 413.
White, M. L. 1949, *Ap. J.*, **109**, 159.
Woolley, R. v. d. R. 1954, *M. N.*, **114**, 191.
—— . and Robertson D. A. 1956, *M. N.*, **116**, 288.
Zeipel, H. von 1913, *Kung. Svenska Vet. Akad. Handlinger*, Bd. 51, No. 5, 9.

Added in proof.

Before eq.(17) at p. 48, the following lines should be inserted:

There is an alternative method to investigate the change of the cluster structure by solving a group of equations derived from eq.(16). Let $f(|v|=v_e)=0$, and β', β'', and β''' be certain approximate mean values of β, and put