APPROXIMATE BOUNDARY LAYER ON THE β-PLANE WITH SPECIAL APPLICATION TO WESTERN BOUNDARY CURRENT

Kiheiji Ogawa

Received August 27, 1690

Abstract

Approximate theory of the boundary layer is extended to the β-plane for homogeneous barotropic flow with examples of special flow pattern.

1. Introduction

After the theoretical finding of the westward intensification of an ocean currents by Stommel (1948), the viscous theory of the wind-driven ocean circulation was completed by Munk (1950), Hidaka (1949) and others. Also, it must be noted that Munk and Carrier (1950) already used the boundary layer technique in their viscous theory of ocean circulation. Later on, however, Stommel suggested that the dynamics of the Gulf Stream will be rather of a non-linear inertial character because of the observed smaller value of the coefficient of lateral eddy viscosity which is by one order smaller than $10^7 \sim 10^8$ adopted by Munk and Hidaka. This inertial model was followed in a mathematically complete form in homogeneous and two-layer models by Charney (1955) and Morgan (1956), use being made of the boundary layer technique.

The aim of the present paper is to investigate the structure of the western boundary current by modelling a visco-inertial homogeneous flow with a technique of the approximate theory of the boundary layer introduced by Pohlhausen (1921). As is pointed out in the papers by Charney and Morgan, separation of the Gulf Stream seems to be due to the baroclinicity of the ocean. On the other hand, as the usual boundary layer theory suggests, non-linearity of dynamical equations will induce the variation of the boundary layer thickness, and it might even lead to the separation of the stream. This paper outlines the approximate boundary layer theory on the β-plane. Numerical examples for the variation of the boundary layer are left to the future.

2. General Scheme of the Approximate Solution of the Boundary Layer

Dynamical equations for a homogeneous barotropic flow on the β-plane are given by

$$
\begin{aligned}
u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}&=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+(f_0+\beta y)v,\\
u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}&=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)-(f_0+\beta y)u,\\
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}&=0,
\end{aligned}
\tag{1}
$$

where x-axis is taken eastward positive, y-axis northward positive, u the flow velocity in the x-direction, v that in the y-direction, ν the coefficient of lateral eddy viscosity, and the Coriolis parameter f is approximated by a linear function of $y: f=f_0+\beta y$. Coordinate axes are shown in Fig. 1.

Regarding the western boundary currents as a boundary layer, we subdivide the ocean circulation into the following two regimes: an interior region, where non-linear inertia terms as well as the viscous terms can be ignored, and a boundary region (Fig. 1).

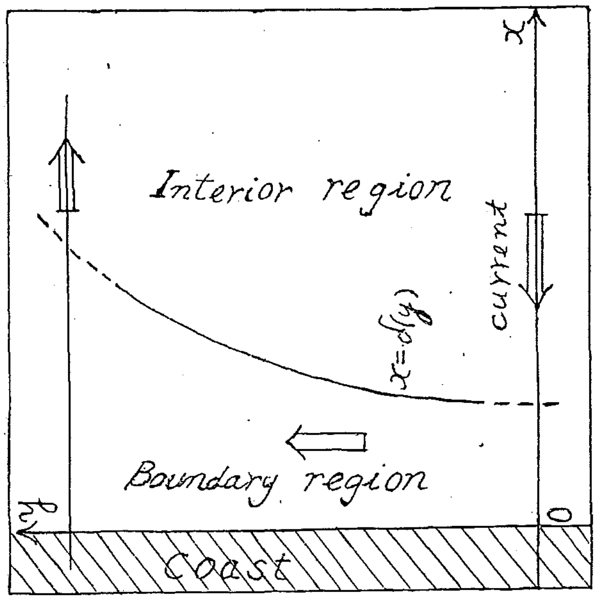

Fig. 1. Interior and boundary regions.

To make a boundary layer approximation, we follow a usual procedure of introducting a characteristic velocity maximum V_m, a characteristic scale length L and non-dimensional variable as follows:

$\bar{u}=u/V_m$, $\bar{v}=v/V_m$, $\bar{x}=x/L$, $\bar{y}=y/L$, $\bar{p}=\dfrac{p}{\rho V_m^2}$.

Here, V_m and L are selected such that $\partial\bar{v}/\partial\bar{y}\leq 1$ (For instance, see Schlichting (1955)).

Then, dropping the bar, equation (1) reduces to the following non-dimensional form

$$
\begin{aligned}
&\underset{\delta\cdot 1}{u\frac{\partial u}{\partial x}}+\underset{1\cdot\delta}{v\frac{\partial u}{\partial y}}=-\underset{1/\delta}{\frac{\partial p}{\partial x}}+\frac{1}{R}\left(\underset{1/\delta}{\frac{\partial^2 u}{\partial x^2}}+\underset{\delta}{\frac{\partial^2 u}{\partial y^2}}\right)+\underset{1}{\Omega v}+\underset{1}{Byv},\\
&\underset{\delta\cdot 1/\delta}{u\frac{\partial v}{\partial x}}+\underset{1\cdot 1}{v\frac{\partial v}{\partial y}}=-\underset{1}{\frac{\partial p}{\partial y}}+\frac{1}{R}\left(\underset{1/\delta^2}{\frac{\partial^2 v}{\partial x^2}}+\underset{1}{\frac{\partial^2 v}{\partial y^3}}\right)+\underset{\delta}{\Omega u}-\underset{\delta}{Byu},
\end{aligned}
\tag{2}
$$

where

$$
R\equiv\frac{V_m L\rho}{\mu}=\frac{V_m L}{\nu},\qquad \Omega\equiv\frac{Lf_0}{\rho V_m},\qquad B\equiv\frac{\beta L^2}{\rho V_m}
$$

Using the estimates

$$
\partial u/\partial x\sim 1,\qquad u\sim\delta,\qquad \partial^2 u/\partial x^2\sim 1/\delta,\qquad \partial v/\partial x\sim 1/\delta \text{ etc,}
$$

where δ is the non-dimensional boundary layer thickness defined by $\delta\equiv l/L \ll 1$, we have the order estimates as listed above in eq. (2).

Hence, in the boundary region, we have the boundary layer equations as follows:

$$0=-\frac{1}{\rho}\frac{\partial p}{\partial x}+(f_0+\beta y)v,$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\frac{\partial^2 v}{\partial x^2}-(f_0+\beta y)u,$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{3}$$

At the outer edge of the boundary layer, we assume that the following approximation is valid:

$$x=\delta(y); \quad u=U, \quad v=V,$$

$$V\frac{\partial V}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}-(f_0+\beta y)U,$$

$$0=-\frac{1}{\rho}\frac{\partial p}{\partial x}+(f_0+\beta y)V,$$

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=0. \tag{4}$$

The current velocities U, V are assumed to have a Sverdrup's solution (1947).

The problem under investigation is to solve eq. (3) which satisfies certain appropriate boundary conditions.

To this end, we adopt a well-known technique of the approximate laminar boundary layer theory extended by Pohlhausen (1921), Holstein and Bohlen (1940) and others, where one deals with the integrated momentum equation with an appropriate velocity profile instead of treating the original dynamical equations (See also Schlitchting (1955), pp. 206—213).

Now, we assume the 4-th power approximation of velocity profile after Pohlhausen as follows:

$$v/V=f(\eta)=a\eta+b\eta^2+c\eta^3+d\eta^4, \qquad 0\leq\eta\leq 1,$$

$$v/V=1, \qquad \eta=1, \tag{5}$$

where $\eta=x/\delta(y)$ and $\delta(y)$ is the thickness of the boundary layer. From the first of eq (3), we have by integrating with respect to x

$$\frac{p}{\rho}=(f_0+\beta y)\int^x v\,dx.$$

Then

$$\frac{1}{\rho}\frac{\partial p}{\partial y}=\beta\int_\delta^x v\,dx+(f_0+\beta y)\int_\delta^x\frac{\partial v}{\partial y}dx+F(x,y).$$

Hence, from the continuity of $\partial p/\partial y$ at $x=\delta$ and eq. (4) we have

$$\frac{1}{\rho}\frac{\partial p}{\partial y}=\beta\int_\delta^x v\,dx-(f_0+\beta y)(ux)-V\frac{\partial V}{\partial y}. \tag{6}$$

Now, the appropriate boundary conditions to determine the velocity profile will be

$$x=0\,; \quad v=0, \quad \nu\frac{\partial^2 v}{\partial x^2}\Big| = -\beta\int_0^\delta v dx,$$
$$x=\delta\,; \quad v=V, \quad \frac{\partial v}{\partial x}=0, \quad \frac{\partial^2 v}{\partial x^2}=S(y, U). \tag{7}$$

Here, the last condition of the 1st row in eq. (7) is obtained by setting $u=v=0$ in the second of eq. (3), $\frac{1}{\rho}\frac{\partial p}{\partial y}$ being replaced by eq. (6). The function $S(y, U)$ is to be determined to satisfy the condition of continuity of u at $x=\delta$ and will be discussed later.

From the above conditions we have the coefficients of eq. (5) as follows

$$\begin{aligned}
a&=\frac{12+\varLambda+\frac{3}{5}L+\varPsi+\frac{\varPsi L}{60}}{6-\frac{L}{20}}, \\
b&=-\frac{3\varLambda+\frac{21}{10}L+\frac{3}{40}\varPsi L}{6-\frac{L}{20}}, \\
c&=\frac{-12+3\varLambda+\frac{11}{5}L-3\varPsi+\frac{\varPsi L}{10}}{6-\frac{L}{20}}, \\
d&=\frac{6-\varLambda-\frac{3}{4}L+2\varPsi-\frac{\varPsi L}{24}}{6-\frac{L}{20}},
\end{aligned} \tag{8}$$

where

$$L=\frac{\beta\delta^3}{\nu}, \quad \varLambda=\frac{\delta^2}{\nu}\frac{dV}{dy} \quad \text{and} \quad \varPsi=\frac{\delta^2}{V}S.$$

Hence we have the velocity profile in the following from

$$\frac{v}{V}=f(\eta)=\frac{1}{1-\frac{L}{120}}\{F(\eta)+\varLambda G(\eta)+LH(\eta)+\varPsi I(\eta)+\varPsi LJ(\eta)\}, \tag{9}$$

where

$$\begin{aligned}
F(\eta)&=2\eta-2\eta^3+\eta^4 \\
G(\eta)&=\frac{1}{6}(\eta-3\eta^2+3\eta^3-\eta^4) \\
H(\eta)&=\frac{1}{120}(12\eta-42\eta^2+44\eta^3-15\eta^4) \\
I(\eta)&=\frac{1}{6}(\eta-3\eta^3+2\eta^4) \\
J(\eta)&=\frac{1}{720}(2\eta-9\eta^2+12\eta^3-5\eta^4)
\end{aligned}$$

Now, using the relation (6), we have the integrated momentum equation in the following form:

$$\frac{d}{dy}[\theta V^2]+\frac{dV}{dy}\delta^* V=\frac{\tau_0}{\rho}+\beta\int_0^\delta\left(\int_\delta^x v dx\right)dx, \tag{10}$$

where

$$\delta^* V=\int_0^\delta (V-v)dx,$$

$$\theta V^2=\int_0^\delta (V-v)v dx, \tag{11}$$

and

$$\tau_0=\mu\frac{\partial v}{\partial x}\bigg|_{x=0}.$$

δ^* and θ defined above are respectively displacement thickness and momentum thickness in the usual boundary layer theory. Equation (10) is a well-known momentum equation except for the last term of the right-hand side which stemms from the effect of the planetary vorticity βv (For the details to derive the momentum equation, see Schlichting, p. 124).

Individual term in eq (10) is readily expressed by the unknown parameters L and S using the relations (9) and (11), hence eq (10) determines a differential relation between L and S: $D_1(L, S)=0$.

Now, as stated above, S is also restricted from the condition of continuity of flow velocity u at the outer edge of the boundary layer. Integration of the equation of continuity from 0 to $\delta(y)$ with respect to x and eq (5) give the following relation

$$U(y)+\frac{d}{dy}\left[\delta.V\int_0^1 f(\eta)d\eta\right]-V\frac{d\delta}{dy}=0. \tag{12}$$

This relation gives together with the relation (9) another differential relation between L and S: $D_2(L, S)=0$.

Thus the problem is now reduced to the following general scheme: determine L and S from the relation $D_1(L, S)=0$ and $D_2(L, S)=0$ and from appropriate physical conditions, then the approximate structure of the western boundary current is completely determined. To solve these two systems of non-linear differential equations in general would be troublesome, and in the next section examples will be given for the special flows such that $S=0$.

3. Boundary Layer Flow for the Special Case $S\equiv\frac{\partial^2 v}{\partial x^2}\Big|_{x=\delta}=0$.

To avoid mathematical complexities of the problem we deal with a boundary layer flow for the special case $S=0$.

Also, to make order eestimates of L and Λ, we assume the constants as follows: $\beta\sim 2\times 10^{-13}/\mathrm{cm\ sec}$, $\delta\sim 10^7$, $s\sim 2\times 10^8$ and $V\sim 1$. Then $\frac{dV}{dy}\sim\frac{V}{s}\sim 5\times 10^{-9}$

$\Lambda \sim \frac{5\times10^5}{\nu}$ and $L \sim \frac{2\times10^8}{\nu}$. Hence, $L \gg \Lambda$ and we set $\Lambda \approx 0$ hereafter since Λ is negligible compared to L.

In this case, a velocity profile is given by setting $\Psi=\Lambda=0$ in eq (9) as follows:

$$v/V=\frac{1}{1-\frac{L}{120}}\{F(\eta)+LH(\eta)\}. \tag{13}$$

Curves of functions $F(\eta)$ and $H(\eta)$ are shown in Fig. 2 together with the functions $G(\eta)$, $I(\eta)$ and $J(\eta)$.

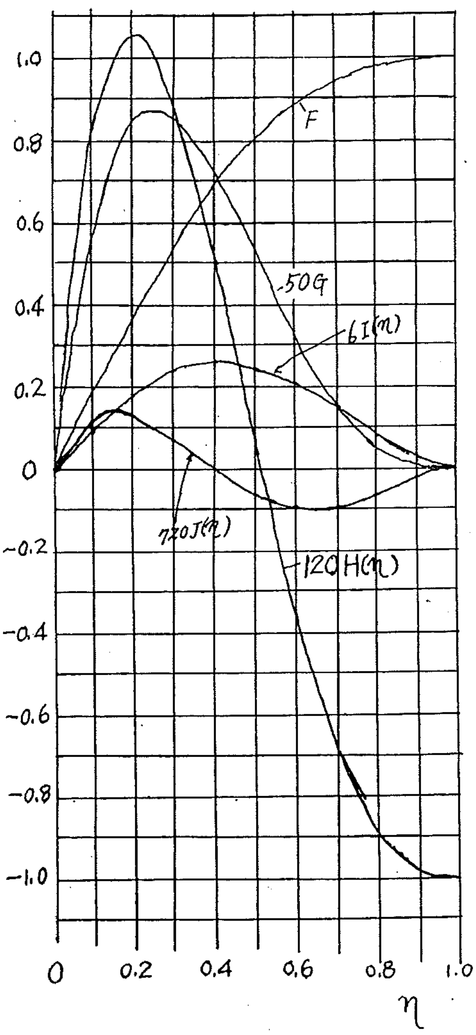

Fig. 2. Curves of F, G, H, I and J

Now, from eqs. (11) and (13) we have the following relations:

$$\frac{\delta^*}{\delta}=\frac{\frac{3}{10}-\frac{L}{120}}{1-\frac{L}{120}},$$

$$\frac{\theta}{\delta}=\frac{1}{\left(1-\frac{L}{120}\right)^2}\left\{\frac{37}{315}-\frac{17}{6300}L-\frac{L^2}{25200}\right\},$$

$$\frac{\tau_0\delta}{\mu V}=\frac{1}{1-\frac{L}{120}}\left\{2+\frac{L}{10}\right\},$$

$$\int_0^\delta\left(\int_\delta^x v dx\right)dx=\frac{\delta^2 V}{1-\frac{L}{120}}\left\{-\frac{13}{30}+\frac{L}{600}\right\}. \tag{14}$$

Then substituting eq (14) into eq (10) and using the relation $\delta=\left(\frac{\nu L}{\beta}\right)^{1/3}$ we have the equation for L such that

$$\underbrace{\frac{dL}{dy}}_{\text{non-linear}}=\frac{\nu^{1/3}\beta^{2/3}}{V}\cdot\frac{C(L)}{g(L)}\left\{-\underbrace{\frac{dV/dy}{\nu^{1/3}\beta^{2/3}}f(L)}_{\text{non-linear}}+\underbrace{A(L)}_{\text{viscous}}+\underbrace{B(L)}_{\text{planetary}}\right\}, \tag{15}$$

where

$$f(L)=\frac{337}{630}-\frac{409}{25200}L-\frac{1}{100800}L^2,$$

$$g(L)=\frac{37}{945}-\frac{223}{113400}L-\frac{61}{567000}L^2+\frac{L^3}{9072000},$$

$$A(L)=L^{-2/3}\left(1-\frac{L}{120}\right)\left(2+\frac{L}{10}\right), \tag{16}$$

$$B(L)=\left(1-\frac{L}{120}\right)\left(-\frac{13}{30}+\frac{L}{600}\right)L^{1/3},$$

$$C(L)=L\left(1-\frac{L}{120}\right).$$

Physical processes corresponding to individual terms eq. (15) are also described below them in the same equation.

Equation (15) has a singularity at $y=y_0$ where velocity $V(y)$ vanishes. From physical consideration, dL/dy must be finitely determined and hence the bracket in eq (15) must vanish at $y=y_0$ if the boundary layer approximation holds there. This case gives the initial values of L and dL/dy and the integration of L is then performed numerically without difficulty.

From eq (14), we see that $L_c=120$ is a singluar point. Also, from Fig. 2 and eq (13), we see that the familiar westward intensification occurs only when $120<L\lesssim 500$. $L_s\approx 1000$ is also a singular point of L-equation since the function $g(L)$ which appears in the denominator of eq. (15) vanishes there.

Considering the above, we seek from now on the solution within the range $L_c<L<L_s$.

Also, from τ_0-equation in eq (14) we see that there is not a separation point $(\partial v/\partial x)_{x=0}=0$, or a counter current near the coastal boundary in this case.

In the following, examples will be given for two cases of wind stress distributions: (a) parabolic wind distribution and (b) sine wind distribution.

(a) Parabolic wind distribution.

Assume that the wind stress is given by

$$\tau_x=-\tau_0\left(1-\frac{y^2}{s^2}\right),\quad \tau_y=0 \text{ for } 0\leq y\leq s$$

Then, the mass transport M_y in the y-direction in the interior region is given by

$$M_y=+\frac{\mathrm{curl}\,\tau}{\beta}=-\frac{2y\tau_0}{\rho\beta s^2},\tag{17}$$

and the flow velocity V on the outer edge of the boundary layer is reduced to the following:

$$V=\frac{M_y}{h}=-V_0\,\xi,\quad \xi=\frac{y}{s},$$

where $V_0=\dfrac{2\tau_0}{\rho\beta sh}$ and h is the depth of no motion assumed constant.

Hence, equation for L is reduced to

$$\frac{dL}{d\xi}=-\frac{K}{\xi}\frac{C(L)}{g(L)}\left\{\frac{1}{K}f(L)+A(L)+B(L)\right\}\tag{18}$$

where

$$K=\frac{\beta^{2/3}\nu^{1/3}s}{V_0}.$$

Here, we remark that the boundary layer and hence, the boundary layer flow is completely determined by a parameter K.

As stated before, initial value L should be chosen so that $dL/d\xi$ may be finitely determined at the initial singular point $\xi=0$ where velocity $V(\xi)$ vanishes. Then, equation for L_0 is

$$\frac{1}{K}f(L)+A(L)+B(L)=0. \tag{18}$$

Also, it is easy to see that the initial value of $dL/d\xi$ at the initial singularity is zero and L is everywhere constant equal to L_0.

Flow velocity in the x-direction at the outer edge of the boundary layer is obtained from eqs (12) and (13) as follows:

$$\frac{U(y)}{\nu^{1/3}\beta^{-1/3}}=-\frac{7}{10}\frac{dV}{dy}\frac{L^{1/3}}{1-\frac{L}{120}}. \tag{20}$$

On the other hand, Sverdrup solution for the velocity U at the outer edge of the boundary layer is given by

$$U\cong-\frac{V_0}{s}r, \tag{21}$$

where r is the distance between east- and west-boundaries in the ocean.

From eq (20) and (21), r is expressed by

$$r\cong-\frac{7}{10}\nu^{1/3}\beta^{-1/3}\frac{L^{1/3}}{1-\frac{L}{120}}, \tag{22}$$

To give numerical examples, we assume $\tau_0\approx1$, $\beta\approx2\times10^{-13}$, $s\approx2\times10^{8}$, $h\approx10^5$, $\nu\approx10^6$, then we have $V_0\approx0.5$ and $K\approx140$. In the following L_0 and r are determined from eqs (19) and (22) for parameters $K\approx100$, 120 and 140.*

K	L_0	r
100	123.50	2000 km
120	122.90	2500 km
140	122.45	2900 km

These flows given above are special examples of the visco-inertial boundary layer flow for the special case $S\equiv\left.\frac{\partial^2 v}{\partial x^2}\right|_{x=\delta}=0$. Here we notice that the ocean width r in the x-direction is one half or one third of the actual ocean flow and is very small. This suggests that in the actual ocean flow we

* L_0 is determined in the visco-inertial-planetary range in eq. (19) in the sense that the individual term of that equation is of comparable order.

must take into account the effect of the function $S \equiv \left.\frac{\partial^2 v}{\partial x^2}\right|_{x=\delta}$.

(b) Sine wind distribution.

Now we assume that the wind stress components are given by

$$\tau_x = -\tau_0 \cos ny, \quad \tau_y = 0, \quad n = \pi/s.$$

Then the flow velocity V is given by

$$V = -V_0 \sin ny, \tag{23}$$

where $$V_0 = \frac{\tau_0 n}{\beta h}.$$

Then, L-equation (15) reduces to the following:

$$\frac{dL}{d\xi} = -\frac{K_s}{\sin \pi\xi} \frac{C(L)}{g(L)} \left[\frac{\pi}{K_s} \cos \pi\xi f(L) + A(L) + B(L) \right], \tag{24}$$

where $$K_s = \frac{\beta^{2/3} \nu^{1/3} s}{V_0}.$$

As numerical constants we assume that $\beta \approx 2 \times 10^{-13}$, $\tau_0 \approx 1$, $s \approx 4 \times 10^8$, $h \approx 5 \times 10^4$, $\nu \approx 10^6$. Then we have $V_0 \approx 0.8$ and $K \approx 173$.

From the same reasoning as stated before, the initial value L_0 at the singularity point $\xi = 0$ of L-equation is obtained from the equation

$$\frac{\pi}{K_s} f(L) + A(L) + B(L) = 0. \tag{25}$$

This equation has two roots in the range $120 < L < 1000$ such that

$$L_0 \approx 184 \quad \text{and} \quad 126.90.$$

Now, from $\left.\frac{d^2 V}{dy^2}\right|_{x=0} = 0$ and from eq (25), we have the initial value $\left.\frac{dL}{d\xi}\right|_{\xi=0} = 0$ and the integration of L -equation is easily performed numerically. The solution L is shown in Fig. 3.

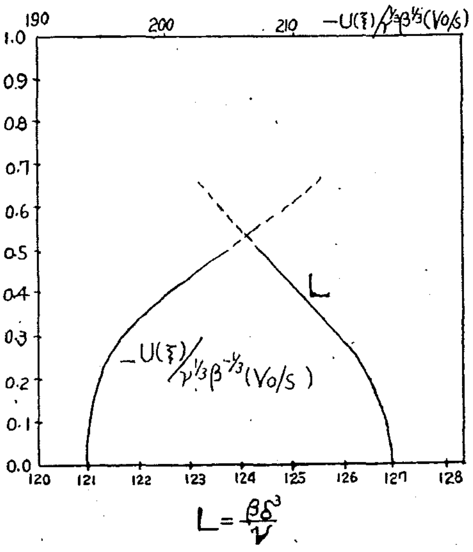

Fig. 3. Curves of L and U

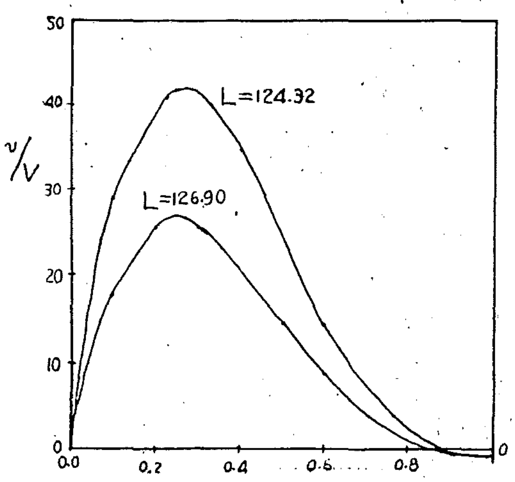

Fig. 4. Curves of $-v/V$

Flow velocity U in the x-direction is computed from eqs (12), (13), (15) and Fig. 3 and is shown also in Fig. 3 for $L_0=126.90$. Also, velocity v in the boundary layer is shown in Fig. 4.

Since $\nu^{1/3}\beta^{-1/3}\approx 1.7\times 10^6$ and $\dfrac{U(y)}{\nu^{1/3}\beta^{-1/3}(V_0/s)}\approx -200$, the ocean width r is about 3400 km, and we see that in this case too the ocean width is about one half smaller compared to the actual ocean.

4. Boundary Layer Flow in the General Case

So far, we have treated the problem with a simplest assumption that $S\equiv\left.\dfrac{\partial^2 v}{\partial x^2}\right|_{x=\delta}\equiv 0$. As one sees there, it gives too small value of an ocean width r of about 3,000 km in the east-west direction, and we need to extend the preceding theory to the general case where the ocean width r is arbitrarily chosen.

In this case the velocity profile is given by eq (9). Also curves of F, G, H, I and J are shown in Fig. 3.

Now, from eqs. (9) and (11) we have the following relations:

$$
\begin{aligned}
\frac{\delta^*}{\delta}&=\frac{\dfrac{3}{10}-\dfrac{L}{120}-\dfrac{\Psi}{40}}{1-\dfrac{L}{120}}, \\
\frac{\theta}{\delta}&=\frac{M(L,\Psi)}{\left(1-\dfrac{L}{120}\right)^2}, \\
\frac{\tau_0\delta}{\mu V}&=\frac{1}{1-\dfrac{L}{120}}\left\{2+\frac{L}{10}+\frac{\Psi}{6}+\frac{L\Psi}{360}\right\}, \\
\int_0^\delta\left(\int_\delta^x v\,dx\right)dx&=\frac{V\delta^2}{1-\dfrac{L}{120}}\left[-\frac{13}{30}+\frac{L}{600}-\frac{\Psi}{90}+\frac{L\Psi}{43200}\right],
\end{aligned}
\tag{26}
$$

where

$$
\begin{aligned}
M(L,\Psi)=&\frac{37}{315}-\frac{37}{3780}\Psi-\frac{17}{6300}L-\frac{71}{453600}L\Psi-\frac{L^2}{25200}-\frac{19}{22680}\Psi^2-\frac{L^2\Psi^2}{81648000}\\
&-\frac{L^2\Psi}{907200}+\frac{L\Psi^2}{5443200}.
\end{aligned}
\tag{27}
$$

Substituting eqs (26) into the integrated momentum equation (10), we have the following equation for L and Ψ:

$$
\underbrace{V\left[\frac{M}{60}+\left(1-\frac{L}{120}\right)\left(M+\frac{M}{3L}\right)\right]\frac{dL}{dy}}_{\text{non-linear}}+\underbrace{\left(1-\frac{L}{120}\right)VM_\Psi\frac{d\Psi}{dy}}_{\text{non-linear}}
$$

$$
=-2\frac{dV}{dy}\left(1-\frac{L}{120}\right)M \quad \cdots\cdots\text{non-linear}
$$

$$+\left(1-\frac{L}{120}\right)^2\left\{-\frac{dV}{dy}\left(\frac{3}{10}-\frac{L}{120}-\frac{\Psi}{40}\right)\right. \quad \cdots\cdots\text{non-linear}$$

$$+\nu^{1/3}\beta^{2/3}L^{-2/3}\left(2+\frac{L}{10}+\frac{\Psi}{6}+\frac{L\Psi}{360}\right) \quad \cdots\cdots\text{viscosity}$$

$$\left.+\nu^{1/3}\beta^{2/3}L^{1/3}\left(-\frac{13}{30}+\frac{L}{600}-\frac{\Psi}{90}+\frac{L\Psi}{43200}\right)\right\} \cdots\text{planetary vorticity}, \qquad (28)$$

where M_L or M_Ψ denotes a partial differentiation of M with respect to L or Ψ. Also, substitution of eqs (26) into the integrated continuity equation (12) gives the following relation:

$$\left[\frac{1}{3L}\left(1-\frac{L}{120}\right)\left(-\frac{3}{10}+\frac{\Psi}{40}+\frac{L}{120}\right)+\frac{1}{120}\left(\frac{7}{10}+\frac{\Psi}{40}\right)\right]\frac{dL}{dy}+\frac{1}{40}\left(1-\frac{L}{120}\right)\frac{d\Psi}{dy}$$

$$=-\left(1-\frac{L}{120}\right)^2\frac{\nu^{-1/3}\beta^{1/3}L^{-1/3}U(y)}{V(y)}-\left(1-\frac{L}{120}\right)\left(\frac{7}{10}+\frac{\Psi}{40}\right)\frac{1}{V}\frac{dV}{dy}. \qquad (29)$$

Eqs (28) and (29) are the fundamental simultaneous differential equations of L and Ψ for the present problem, the solution of which gives the complete understanding of the approximate boundary layer.

Next, as an example solution will be sought for the parabolic wind distribution.

As shown in Sec. 3 (a), the flow velocities at the outer edge of the boundary layer are given by

$$U=-\frac{V_0}{s}\gamma, \qquad V=-V_0\xi, \qquad \xi=\frac{y}{s},$$

where

$$V_0=\frac{2\tau_0}{\rho\beta sh}.$$

As we readily see, the inflowing edge $y=0$ of the current into the boundary layer forms a singular point of L, Ψ-equations since the velocity V vanishes there. If the flow forms the boundary layer even at this singularity, then we seek the sloution which gives the finitely determined values of $\frac{dL}{dy}$ and $\frac{d\Psi}{dy}$ from the physical point of view as was discussed in the preceding section.

Then eqs. to determine the initial values of L and Ψ are given by

$$2M=\left(\frac{L}{120}-1\right)\{\alpha(L)+K[L^{-2/3}\beta(L)+L^{1/3}\gamma(L)\},$$

$$\frac{\Psi}{40}=A\left(\frac{L}{120}-1\right)L^{-1/3}-\frac{7}{10}, \qquad (30)$$

where

$$A=\nu^{-1/3}\beta^{1/3}\gamma, \qquad K=\frac{\nu^{1/3}\beta^{1/3}s}{V_0},$$

and

$$\alpha(L)=\frac{3}{10}-\frac{L}{120}-\frac{\Psi}{40},$$
$$\beta(L)=2+\frac{L}{10}+\frac{\Psi}{6}+\frac{L\Psi}{360},$$
$$\gamma(L)=-\frac{13}{30}+\frac{L}{600}-\frac{\Psi}{90}+\frac{L\Psi}{43200}.$$

If one finds unique solutions for L and Ψ from eq (30), then L and Ψ are constant throughout all y even at the inflowing edge $y=0$.

For the parameters $\nu=10^6$, $\beta=2\times10^{-13}$, $r=10000\,\mathrm{km}$, we cannot find the unique solution of eq (30) even at relatively larger value of K*. This probably means that the boundary layer approximation no longer holds in the vicinity of the inflowing edge $y=0$ of the current. This may suggest that in the actual ocean circulation we must adopt another model rather than the boundary layer one in the vicinity.

5. Concluding Remarks

As is well known, the structure of the Western Boundary Currents such as the Kuroshio and the Gulf Stream is not so simple as shown by the theory of Munk and Hidaka. Charney (1955) and Morgan (1956) suggest the important role of the baroclinicity of the ocean on the separation of the Gulf Stream from the boundary.

Now apart from the baroclinicity of the ocean, non-linearity of the dynamical equations may alter the boundary layer thickness as the usual boundary layer theory suggests, and there exists a possibility that it might lead to the separation of the stream from the coast.

As was suggested in the preceding section, inflowing edge $y=0$ of the boundary layer in the case of the sine wind distribution is not in the boundary layer in the usual circumstances. So, to compute the variation of the boundary layer, we integrate eqs (28) and (29) with adequately prescribed initial values of L and Ψ at the initial boundary layer which may be determined from the observed data.

Next, it must be remarked that the leaving point of the flow from the boundary layer is another singular point of the L, Ψ-equations since there velocity V vanishes and the boundary layer approximation no longer holds.

Numerical examples on the boundary layer variation are left to the future.

Appendix

Table for velocity profile for various values of L is obtained through the Runge's method of integration.

---

* We cannot find the roots of eq (30) for the parameter $K=140$.
For much larger values of K, there seems to be a possibility that a unique solution could be found.

TABLE OF v/V FOR VARIOUS VALUES OF L

η	L 210	180	160	140	130	
0.0	0.00	0.00	0.00	0.00	0.0	
0.1	−2.71	−2.84	−3.27	−6.90	−13.0	
0.2	−2.96	−3.90	−5.31	−9.60	−18.1	·········max.
0.3	−2.79	−3.74	−5.16	−9.42	−17.9	
0.4	−2.11	−2.92	−4.11	−7.74	−14.9	
0.5	−1.26	−1.86	−2.76	−5.40	−10.8	
0.6	−0.33	−0.68	−1.20	−2.76	−5.9	
0.7	+0.37	+0.20	−0.06	−0.78	−2.3	
0.8	0.77	0.72	+0.63	+0.36	−0.24	
0.9	0.97	0.96	0.96	0.90	+0.84	
1.0	+1.00	+1.00	+1.00	+1.00	+1.00	

References

Charney, J. G. 1955, *Proc. Nat. Acad. Sci. Wash.*, **41**, 731.
Hidaka, K. 1949, *Geophys. Notes, Univ. of Tokyo*, Vol. **2**, No. 3.
″ 1949, *Journ. Mar. Res.*, **8**, 132—136.
Holstein, H. and Bohlen, T. 1940, *Lilienthal-Bericht*, s. 10, 5.
Morgan, G. W. 1956, *Tellus*, **8**, 301.
Munk, W. H. 1950, *Journ. Meteor.*, **7**, 79.
Munk, W. H. and Carrier G. F. 1950, *Tellus*, **2**, 158.
Pohlhausen, K. 1921, *Z. A. M. M.* **1**, 252.
Schlichting, H. 1955, *Boundary Layer Theory*, (Pergamon Press), translated from the German textbook *Grenzschichttheorie*.
Stommel, H. 1948, *Trans. Amer. Geophys. Union*, **29**, 202—206.
Stommel, H. 1958, *The Gulf Stream*, (Univ. of Calif. Press).
Sverdrup, H. U. 1947, *Proc. Nat. Acad. Sci. Wash.*, **33**, 318—326.

Kiheiji OGAWA, M. Sc., 1931—1960,

graduated in March, 1956, from Master Course (geophysics, specially oceanography) in the Faculty of Science, University of Tokyo, and working since April, 1958, at the Hydrographic Office of Japan, suddenly died on September 2, on the voyage to America.

His main works are:

On the seasonal variations of surface divergence of the ocean currents in terms of wind stresses over the ocean (collaborated with K. Hidaka, University of Tokyo), *Records of Oceanographic Works in Japan,* vol. 4, No. 2 (New Series), 1958.

A practical method of a determination of reflection of long gravitational waves, (collaborated with K. Yoshida, University of Tokyo), *ibid.,* vol. 5, No. 1 (New Series), 1959,

Edge waves induced by a radially spreading long wave and its damping due to the irregularity of coast, *Contr. Mar. Res. Lab., H. O.,* **1**, 103, 1960,

and

Approximate boundary layer on the β-plane with special application to western boundary current, *ibid.,* **2**, 73, 1960.