ON A CONDITION FOR SEPARATING ARGUMENTS OF AN IMPLICIT FUNCTION

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Abstract

Necessary and sufficient condition on a function of $N+M$ variables

$$F(x_1, \cdots\cdots, x_M; p_1, \cdots\cdots, p_N)=0$$

for the fact that the arguments can be separated into two groups, or the function may be transformed into a form

$$f(x_1, \cdots\cdots, x_M)=g(p_1, \cdots\cdots, p_N)$$

is given.

1. Introduction. On constructing tables or nomograms for obtaining values of a function with many arguments it is desirable for the function to be transformed into a suitable and convenient form. One of such forms is the one in which the arguments are separated into two groups. If a given function of $N+M$ arguments

$$F(x_1, \cdots\cdots, x_M; p_1, \cdots\cdots, p_N)=0 \quad \cdots\cdots(1)$$

is transformed into a form

$$f(x_1, \cdots\cdots, x_M)=g(p_1, \cdots\cdots, p_N), \quad \cdots\cdots(2)$$

then the required tables or nomograms are reduced to a set of N and M argument tables or nomograms instead of $N+M$ arguments.

However, there exists a strong condition upon the function F for the possibility of the above transformation. Therefore, it seems important to give this condition, otherwise large effort might be paid in vain to make tables or nomograms for computing values of one of the arguments as a function of the other.

In this paper we shall give the necessary and sufficient condition under the assumptions that the functions F, f and g are continuous and differentiable at least up to the second order for each variable, and their first order derivatives are not identically zero.

2. Necessary Condition. Let the function F be transformed into the form (2). Then by the assumptions we can write as follows:

$$p_N=G(x_1, \cdots\cdots, x_M; p_1, \cdots\cdots, p_{N-1}),$$

and

$$p_N=Q[f(x_1, \cdots\cdots, x_M); p_1, \cdots\cdots, p_{N-1}].$$

Because G and Q express the same function of $x_1, \cdots\cdots, x_M; p_1, \cdots\cdots,$ and p_{N-1}, we have*

$$dp_N=\sum_{j=1}^{M} G_{x_j} dx_j+\sum_{i=1}^{N-1} G_{p_i} dp_i=\sum_{j=1}^{M} Q_f f_{x_j} dx_j+\sum_{i=1}^{N-1} Q_{p_i} dp_i,$$

in which

$$G_{p_i}=Q_{p_i}.$$

Since $dx_1, \cdots\cdots, dx_M, dp_1, \cdots\cdots, dp_{N-1}$ are arbitrary and independent to each other, it

* Partial derivative of a function F (say) with respect to a variable x (say) is denoted as F_x for simplicity. Similarly second derivative is denoted as F_{xy}, etc.

must be

$$G_{x_j}=Q_f f_x,$$

and

$$G_{x_j p_i}=Q_{f p_i} f_{x_j}=\frac{Q_{f p_i}}{Q_f} G_{x_j},$$

for any $i=1, 2, \cdots\cdots, N-1$, and $j=1, 2, \cdots\cdots, M$.
Therefore,

$$G_{x_1} : G_{x_2} : \cdots\cdots : G_{x_M}=G_{x_1 p_i} : G_{x_2 p_i} : \cdots\cdots : G_{x_M p_i}, \qquad (3)$$

for any $i=1, 2, \cdots\cdots, N-1$.

Expressing (3) in terms of F, we have finally

$$F_{x_1} : F_{x_2} : \cdots\cdots : F_{x_M}=\begin{vmatrix} F_{p_i} & F_{p_N} \\ F_{x_1 p_i} & F_{x_2 p_N} \end{vmatrix} : \begin{vmatrix} F_{p_i} & F_{p_N} \\ F_{x_2 p_i} & F_{x_2 p_N} \end{vmatrix} : \cdots\cdots : \begin{vmatrix} F_{p_i} & F_{p_N} \\ F_{x_M p_i} & F_{x_M p_N} \end{vmatrix}, \qquad (4)$$

for all $i \leqq N-1$. Thus this is the necessary condition that (4) holds for all $i \leqq N-1$.

3. Sufficient Condition. Let (4) hold identically for all $i \leqq N-1$. Then by inversing the above argument in the preceding section, we see that (3) holds identically for all $i \leqq N-1$. Now we can easily see that any ratio G_{x_r}/G_{x_s} is independent to all $p_1, \cdots\cdots, p_{N-1}$:
If

$$G_{x_r}=R_{rs}(x_1, \cdots\cdots, x_M ; p, \cdots\cdots, p_{N-1})\ G_{x_s}, \qquad (5)$$

then due to (3)

$$G_{x_r p_i}=R_{rs}\ (x_1, \cdots\cdots, x_M ; p_1, \cdots\cdots, p_{N-1})\ G_{x_s p_i}.$$

By differentiating (5) with respect to p_i, we can see clearly that M must not contain p_i, where i takes all the integer smaller than or equal to $N-1$, and that R is independent to $p_1, \cdots\cdots$, and p_{N-1}.

On considering equations

$$\left.\begin{array}{l} G_{x_1}-R_{12}\ G_{x_2}=0, \\ G_{x_2}-R_{23}\ G_{x_3}=0, \\ \cdots\cdots\cdots \\ G_{x_{M-1}}-R_{M-1, M} G_{x_M}=0, \end{array}\right\} \qquad (6)$$

we have general solutions of each partial differential equation for G through Lagrange's characteristic equations:

$$\frac{dx_1}{1}=\frac{dx_2}{-R_{12}}=\frac{dG}{0}, \cdots\cdots, \frac{dx_{M-1}}{1}=\frac{dx_M}{-R_{M-1, M}}=\frac{dG}{0},$$

which give respectively integrals of the forms

$$G=a,$$
$$L_{12}(x_1, \cdots\cdots, x_M)=b, \qquad \text{etc.,}$$

where a and b do not involve x_j's, but may be functions of p_i's.
Thus the general solutions take the form:

$$G=G\ [L_{12}\ (x_1, \cdots\cdots, x_M) ; p_1, \cdots\cdots, p_{N-1}],$$
$$G=G\ [L_{23}\ (x_1, \cdots\cdots, x_M) ; p_1, \cdots\cdots, p_{M-1}],$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$G=G\ [L_{m-1, m}\ (x_1, \cdots\cdots, x_M) ; p_1, \cdots\cdots, p_{N-1}].$$

Now, if G contains more than two functions of $x_1, \cdots\cdots, x_M$, say,

$$G=G\ [L_1, L_2, \cdots\cdots, L_s ; p_1, \cdots\cdots, p_{N-1}],$$

then

$$G_{x_1}=G_{L_1}\ L_{1 x_1}+\cdots\cdots+G_{L_s}\ L_{s x_1},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$G_{x_t} = G_{L_1} L_{x_t} + \cdots\cdots + G_{L_s} L_{sx_t},$$

and with the aid of (6)

$$\left.\begin{array}{l} L_{1x_1} - L_{1x_2} R_{12} = 0, \\ L_{2x_2} - L_{2x_2} R_{12} = 0, \\ \cdots\cdots\cdots \\ L_{sx_1} - L_{sx_2} R_{12} = 0, \end{array}\right\} \qquad \left.\begin{array}{l} L_{1x_2} - L_{1x_3} R_{13} = 0, \\ L_{2x_2} - L_{2x_3} R_{13} = 0, \\ \cdots\cdots\cdots \\ L_{sx_2} - L_{sx_3} R_{13} = 0, \end{array}\right\} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \text{ etc.}$$

but R_{12}, $R_{13}, \cdots\cdots$, $R_{m-1,\, m}$, and L_1, $L_2, \cdots\cdots L_s$ are functions of x_1, $x, \cdots\cdots$, and, x_M only, so that we find readily that

$$L_1(x_1, \cdots\cdots, x_M) \equiv L_2(x_1, \cdots\cdots, x_M) \equiv \cdots\cdots \equiv L_s(x_1, \cdots\cdots, x_M),$$

hence G takes the form

$$G = G[L(x_1, \cdots\cdots, x_M);\, p_1, \cdots\cdots, p_{N-1}].$$

Thus we have proved that the necessary and sufficient condition for the possibility of separation of variables is that the equation (4) holds identically for all $i \leqq N-1$.

As the special case we consider a function of four arguments. In this case (4) reduces to

$$F_{x_1} : F_{x_2} = \begin{vmatrix} F_{p_1} & F_{p_2} \\ F_{x_1 p_1} & F_{x_1 p_2} \end{vmatrix} : \begin{vmatrix} F_{p_1} & F_{p_2} \\ F_{x_2 p_1} & F_{x_2 p_2} \end{vmatrix}, \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (7)$$

which is the result obtained by E. Goursat [1].

The most familiar application of this formula is the proof of impossibility of separation of variables for the equation of altitude of a star in spherical astronomy,

$$\sin a = \sin l \sin d + \cos l \cos d \cos h,$$

where a, h, d, and l denote star's altitude, hour angle, declination, and latitude, respectively. The impossibility of separation of any two variables is obvious from the formula (7). This fact makes constructing altitude tables or nomograms for astronomical navigation very difficult.

Reference

[1] E. Goursat: Bull. de la Soc. math. de France, t. XXVII, p. 27.